ANSWER KEY

1. **A**	6. **A, B**	12. **C**	18. **4**	24. **A**
2. **D**	7. **E**	13. **A**	19. **A**	25. **B**
3. $\frac{1}{4}$	8. **D**	14. **B**	20. **C**	
	9. **490**	15. **B**	21. **D**	
4. **B**	10. **D**	16. **D**	22. **C**	
5. **D**	11. **C**	17. **C**	23. **A**	

Answer Explanations

1. **(A)** Of every 4 employees, 3 are not college graduates, and 1 is a college graduate. So the ratio of graduates to nongraduates is 1:3.

2. **(D)** Cross-multiplying, we get: $2a^2 = 90 \Rightarrow a^2 = 45$.

3. $\frac{1}{4}$ If 80% were rejected, 20% were accepted, and the ratio of accepted to rejected is 20:80 = 1:4.

4. **(B)** Set up a proportion: $\frac{50 \text{ pages}}{1 \text{ hour}} = \frac{50 \text{ pages}}{60 \text{ minutes}} = \frac{x \text{ pages}}{50 \text{ minutes}}$,

   and cross-multiply: $50 \times 50 = 60x \Rightarrow 2500 = 60x \Rightarrow x = 41\frac{2}{3}$.

5. **(D)** Out of every 8 team members, 3 are juniors and 5 are seniors. Seniors, therefore, make up $\frac{5}{8}$ = 62.5% of the team.

6. **(A)(B)** It is worth remembering that if the ratio of the measures of the angles of a triangle is 1:1:2, the angles are 45-45-90 (see Section 11-J). Otherwise, the first step is to write $x + x + 2x = 180 \Rightarrow 4x = 180 \Rightarrow x = 45$.

   Since two of the angles have the same measure, the triangle is isosceles, and since one of the angles measures 90°, it is a right triangle. I and II are true, and, of course, III is false.

7. **(E)** By definition, π is the ratio of the circumference to the diameter of a circle (see Section 11-L). Therefore, $\pi = \frac{C}{d} = \frac{C}{2r} \Rightarrow 2\pi = \frac{C}{r}$.

8. **(D)** The *fraction* of the team that is sophomores is $\frac{7}{4+7+6+8} = \frac{7}{25}$, and $\frac{7}{25} \times 100\% = 28\%$.

9. **490** Let the number of students taking Spanish be $7x$, and the number taking French be $2x$. Then, $2x = 140 \Rightarrow x = 70 \Rightarrow 7x = 490$.

10. **(D)** Since $\frac{a}{b} = \frac{3}{5}$, $\frac{b}{a} = \frac{5}{3}$. So, $b{:}c = \frac{b}{c} = \frac{b}{\not{a}_1} \times \frac{\not{a}^1}{c} = \frac{5}{3} \times \frac{5}{7} = \frac{25}{21} = 25{:}21$.

    Alternatively, we could write equivalent ratios with the same value for a:

    $$a{:}b = 3{:}5 = 15{:}25 \text{ and } a{:}c = 5{:}7 = 15{:}21.$$

    So, when $a = 15$, $b = 25$, and $c = 21$.

11. **(C)** To solve a proportion, cross-multiply: $\frac{x}{3} = \frac{12}{x} \Rightarrow x^2 = 36 \Rightarrow x = 6$.

12. **(C)** Let $b = 7x$ and $a = 2x$. Then, $7x + 2x = 180 \Rightarrow 9x = 180 \Rightarrow x = 20 \Rightarrow b = 140$ and $a = 40 \Rightarrow b - a = 140 - 40 = 100$.

13. **(A)** Set up the proportion, keeping track of units: $\frac{x \text{ feet}}{h \text{ hours}} = \frac{{}^1\not{12}x \text{ inches}}{{}_5\not{60}h \text{ minutes}} = \frac{i \text{ inches}}{m \text{ minutes}} \Rightarrow \frac{x}{5h} = \frac{i}{m} \Rightarrow x = \frac{5hi}{m}$.

14. **(B)** Gilda grades at the rate of $\frac{t \text{ tests}}{\frac{1}{x} \text{hours}} = \frac{tx \text{ tests}}{1 \text{ hour}}$.

    Since she can grade tx tests each hour, in x hours she can grade $x(tx) = tx^2$ tests.

15. **(B)** Suppose that x boys and x girls joined the club. Then, the new ratio of boys to girls would be $(3 + x){:}(5 + x)$, which we are told is 3:4.

    So, $\frac{3+x}{5+x} = \frac{3}{4} \Rightarrow 4(3 + x) = 3(5 + x) \Rightarrow 12 + 4x = 15 + 3x \Rightarrow x = 3$.

    Therefore, 3 boys and 3 girls joined the other 3 boys and 5 girls: a total of 14.

16. **(D)** Cross-multiplying, we get:

    $11(3x - 1) = 25(x + 5) \Rightarrow 33x - 11 = 25x + 125 \Rightarrow 8x = 136 \Rightarrow x = 17$.

17. **(C)** Since 4 boys can shovel the driveway in 2 hours, or 2 × 60 = 120 minutes, the job takes 4 × 120 = 480 boy-minutes; and so 5 boys would need $\frac{480 \text{ boy-minutes}}{5 \text{ boys}} = 96$ minutes.

18. **4** Since 500 pounds will last for 20 pig-weeks = 140 pig-days, 200 pounds will last for $\frac{2}{5} \times 140$ pig-days = 56 pig-days, and $\frac{56 \text{ pig-days}}{14 \text{ pigs}} = 4$ days.

19. **(A)** Assume that to start there were $3x$ red marbles and $5x$ blue ones and that y of each color were added.

	Quantity A	Quantity B
	$\frac{3x+y}{5x+y}$	$\frac{3}{5}$
Cross-multiply:	$5(3x + y)$	$3(5x + y)$
Distribute:	$15x + 5y$	$15x + 3y$
Subtract $15x$:	$5y$	$3y$

Since y is positive, Quantity A is greater.

20. **(C)** The shares are $2x$, $5x$, and $8x$, and their sum is 3000:
$2x + 5x + 8x = 3000 \Rightarrow 15x = 3000 \Rightarrow x = 200$, and so $8x - 2x = 6x = 1200$.

21. **(D)** Ratios alone can't answer the question, "How many?" There could be 5 boys in the chess club or 500. We can't tell.

22. **(C)** Assume that Sally invited x boys and x girls. When she wound up with x girls and $x + 5$ boys, the girl:boy ratio was 4:5. So,
$\frac{x}{x+5} = \frac{4}{5} \Rightarrow 5x = 4x + 20 \Rightarrow x = 20$
Sally invited 40 people (20 boys and 20 girls).

23. **(A)** If the probability of drawing a red marble is $\frac{3}{7}$, 3 out of every 7 marbles are red, and 4 out of every 7 are non-red. So the ratio of red:non-red = 3:4, which is greater than $\frac{1}{2}$.

24. **(A)** Multiplying the first equation by 3 and the second by 2 to get the same coefficient of b, we have: $9a = 6b$ and $6b = 10c$. So, $9a = 10c$ and $\frac{a}{c} = \frac{10}{9}$.

25. **(B)** Assume the radius of circle I is 1 and the radius of circle II is 3. Then the areas are π and 9π, respectively. So, the area of circle II is 9 times the area of circle I, and $3\pi > 9$.

11-E. AVERAGES

The ***average*** of a set of n numbers is the sum of those numbers divided by n.

$$\text{average} = \frac{\text{sum of the } n \text{ numbers}}{n} \quad \text{or simply} \quad A = \frac{\text{sum}}{n}.$$

If the weights of three children are 80, 90, and 76 pounds, respectively, to calculate the average weight of the children, you would add the three weights and divide by 3:

$$\frac{80 + 90 + 76}{3} = \frac{246}{3} = 82$$

The technical name for this type of average is "arithmetic mean," and on the GRE those words always appear in parentheses—for example, "What is the average (arithmetic mean) of 80, 90, and 76?"

Usually, on the GRE, you are not asked to find an average; rather, you are given the average of a set of numbers and asked for some other information. The key to solving all of these problems is to first find the sum of the numbers. Since $A = \frac{\text{sum}}{n}$, multiplying both sides by n yields the equation: sum = nA.

TACTIC E1

If you know the average, A, of a set of n numbers, multiply A by n to get their sum.

EXAMPLE 1

One day a supermarket received a delivery of 25 frozen turkeys. If the average (arithmetic mean) weight of a turkey was 14.2 pounds, what was the total weight, in pounds, of all the turkeys?

SOLUTION.
Use TACTIC E1: $25 \times 14.2 =$ **355**.

NOTE: We do not know how much any individual turkey weighed nor how many turkeys weighed more or less than 14.2 pounds. All we know is their total weight.

EXAMPLE 2

Sheila took five chemistry tests during the semester and the average (arithmetic mean) of her test scores was 85. If her average after the first three tests was 83, what was the average of her fourth and fifth tests?

Ⓐ 83 Ⓑ 85 Ⓒ 87 Ⓓ 88 Ⓔ 90

SOLUTION.
- Use TACTIC E1: On her five tests, Sheila earned $5 \times 85 = 425$ points.
- Use TACTIC E1 again: On her first three tests she earned $3 \times 83 = 249$ points.

- Subtract: On her last two tests Sheila earned 425 – 249 = 176 points.
- Calculate her average on her last two tests: $\frac{176}{2}$ = **88 (D)**.

NOTE: We cannot determine Sheila's grade on even one of the tests.

KEY FACT E1

- **If all the numbers in a set are the same, then that number is the average.**
- **If the numbers in a set are not all the same, then the average must be greater than the smallest number and less than the largest number. Equivalently, at least one of the numbers is less than the average and at least one is greater.**

If Jessica's test grades are 85, 85, 85, and 85, her average is 85. If Gary's test grades are 76, 83, 88, and 88, his average must be greater than 76 and less than 88. What can we conclude if, after taking five tests, Kristen's average is 90? We know that she earned exactly 5 × 90 = 450 points, and that either she got a 90 on every test or at least one grade was less than 90 and at least one was over 90. Here are a few of the thousands of possibilities for Kristen's grades:

(a) 90, 90, 90, 90, 90 (b) 80, 90, 90, 90, 100 (c) 83, 84, 87, 97, 99
(d) 77, 88, 93, 95, 97 (e) 50, 100, 100, 100, 100

In (b), 80, the one grade below 90, is *10 points below*, and 100, the one grade above 90, is *10 points above*. In (c), 83 is 7 points below 90, 84 is 6 points below 90, and 87 is 3 points below 90, for a total of 7 + 6 + 3 = *16 points below 90*; 97 is 7 points above 90, and 99 is 9 points above 90, for a total of 7 + 9 = *16 points above 90*.

These differences from the average are called ***deviations***, and the situation in these examples is not a coincidence.

KEY FACT E2

The total deviation below the average is equal to the total deviation above the average.

EXAMPLE 3

If the average (arithmetic mean) of 25, 31, and x is 37, what is the value of x?

SOLUTION 1.
Use KEY FACT E2. Since 25 is 12 less than 37 and 31 is 6 less than 37, the total deviation below the average is 12 + 6 = 18. Therefore, the total deviation above must also be 18. So, x = 37 + 18 = **55**.

SOLUTION 2.
Use TACTIC E1. Since the average of the three numbers is 37, the sum of the 3 numbers is 3 × 37 = 111. Then,

$$25 + 31 + x = 111 \Rightarrow 56 + x = 111 \Rightarrow x = \mathbf{55}.$$

KEY FACT E3

Assume that the average of a set of numbers is *A*. If a number *x* is added to the set and a new average is calculated, then the new average will be less than, equal to, or greater than *A*, depending on whether *x* is less than, equal to, or greater than *A*, respectively.

EXAMPLE 4

Quantity A	Quantity B
The average (arithmetic mean) of the integers from 0 to 12	The average (arithmetic mean) of the integers from 1 to 12

TIP
Remember TACTIC 5 from Chapter 9. We don't have to *calculate* the averages, we just have to *compare* them.

SOLUTION 1.
Quantity B is the average of the integers from 1 to 12, which is surely greater than 1. Quantity A is the average of those same 12 numbers and 0. Since the extra number, 0, is less than Quantity B, Quantity A must be *less* [KEY FACT E3]. The answer is **B**.

SOLUTION 2.

Clearly the sum of the 13 integers from 0 to 12 is the same as the sum of the 12 integers from 1 to 12. Since that sum is positive, dividing by 13 yields a smaller quotient than dividing by 12 [KEY FACT B4].

Although in solving Example 4 we didn't calculate the averages, we could have:

$0 + 1 + 2 + 3 + 4 + 5 + \mathbf{6} + 7 + 8 + 9 + 10 + 11 + 12 = 78$ and $\frac{78}{13} = 6.$

$1 + 2 + 3 + 4 + 5 + \mathbf{6} + \mathbf{7} + 8 + 9 + 10 + 11 + 12 = 78$ and $\frac{78}{12} = 6.5.$

Notice that the average of the 13 *consecutive* integers 0, 1,...,12 is the *middle integer*, **6**, and the average of the 12 *consecutive* integers 1, 2,...,12 is the *average of the two middle integers*, **6** and **7**. This is a special case of KEY FACT E4.

KEY FACT E4

Whenever *n* numbers form an arithmetic sequence (one in which the difference between any two consecutive terms is the same): (i) if *n* is odd, the average of the numbers is the middle term in the sequence and (ii) if *n* is even, the average of the numbers is the average of the two middle terms, which is the same as the average of the first and last terms.

For example, in the arithmetic sequence 6, 9, 12, 15, 18, the average is the middle number, **12**; in the sequence 10, 20, 30, 40, 50, 60, the average is **35**, the average of the two middle numbers — 30 and 40. Note that 35 is also the average of the first and last terms—10 and 60.

EXAMPLE 5

On Thursday, 20 of the 25 students in a chemistry class took a test and their average was 80. On Friday, the other 5 students took the test, and their average was 90. What was the average (arithmetic mean) for the entire class?

SOLUTION.
The class average is calculated by dividing the sum of all 25 test grades by 25.

- The first 20 students earned a total of: $20 \times 80 = 1600$ points
- The other 5 students earned a total of: $5 \times 90 = 450$ points
- Add: altogether the class earned: $1600 + 450 = 2050$ points
- Calculate the class average: $\frac{2050}{25} = \mathbf{82}$.

Notice that the answer to Example 5 is *not* 85, which is the average of 80 and 90. This is because the averages of 80 and 90 were earned by different numbers of students, and so the two averages had to be given different weights in the calculation. For this reason, this is called a ***weighted average***.

KEY FACT E5

To calculate the weighted average of a set of numbers, multiply each number in the set by the number of times it appears, add all the products, and divide by the total number of numbers in the set.

So, the solution to Example 5 should look like this:

$$\frac{20(80)+5(90)}{25} = \frac{1600+450}{25} = \frac{2050}{25} = 82$$

Problems involving *average speed* will be discussed in Section 11-H, but we mention them briefly here because they are closely related to problems on weighted averages.

TIP

Without doing any calculations, you should immediately realize that since the grade of 80 is being given more weight than the grade of 90, the average will be closer to 80 than to 90 — certainly *less than 85*.

EXAMPLE 6

For the first 3 hours of his trip, Justin drove at 50 miles per hour. Then, due to construction delays, he drove at only 40 miles per hour for the next 2 hours. What was his average speed, in miles per hour, for the entire trip?

Ⓐ 40 Ⓑ 43 Ⓒ 46 Ⓓ 48 Ⓔ 50

SOLUTION.
This is just a weighted average:

$$\frac{3(50)+2(40)}{5} = \frac{150+80}{5} = \frac{230}{5} = \mathbf{46}.$$

Note that in the fractions above, the numerator is the total distance traveled and the denominator the total time the trip took. This is *always* the way to find an average speed. Consider the following slight variation on Example 6.

EXAMPLE 6A

For the first 100 miles of his trip, Justin drove at 50 miles per hour, and then due to construction delays, he drove at only 40 miles per hour for the next 120 miles. What was his average speed, in miles per hour, for the entire trip?

miles per hour

SOLUTION.
This is not a *weighted* average. Here we immediately know the total distance traveled, 220 miles. To get the total time the trip took, we find the time for each portion and add: the first 100 miles took $100 \div 50 = 2$ hours, and the next 120 miles took $120 \div 40 = 3$ hours. So the average speed was $\frac{220}{5} = \mathbf{44}$ miles per hour.

Notice that in Example 6, since Justin spent more time traveling at 50 miles per hour than at 40 miles per hour, his average speed was closer to 50; in Example 6a, he spent more time driving at 40 miles per hour than at 50 miles per hour, so his average speed was closer to 40.

Two other terms that are associated with averages are ***median*** and ***mode***. In a set of n numbers that are arranged in increasing order, the ***median*** is the middle number (if n is odd), or the average of the two middle numbers (if n is even). The ***mode*** is the number in the set that occurs most often.

EXAMPLE 7

During a 10-day period, Jorge received the following number of phone calls each day: 2, 3, 9, 3, 5, 7, 7, 10, 7, 6. What is the average (arithmetic mean) of the median and mode of this set of data?

Ⓐ 6 Ⓑ 6.25 Ⓒ 6.5 Ⓓ 6.75 Ⓔ 7

SOLUTION.
The first step is to write the data in increasing order:

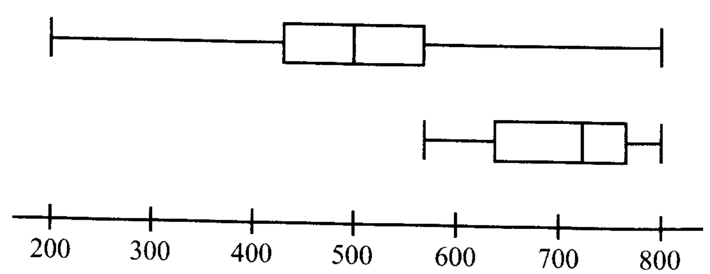
2, 3, 3, 5, 6, 7, 7, 7, 9, 10.

- The median is 6.5, the average of the middle two numbers.
- The mode is 7, the number that appears more times than any other.
- The average of the median and the mode is $\frac{6.5+7}{2} = \mathbf{6.75}$ **(D)**.

The median is actually a special case of a measure called a ***percentile***. In the same way that the median divides a set of data into two roughly equal groups, percentiles divide a set of data into 100 roughly equal groups. P_{63}, the 63rd percentile, for example, is a number with the property that 63% of the data in the group is less than or equal to that number and the rest of the data is greater than that number. Clearly, percentiles are mainly used for large groups of data—it doesn't make much sense to talk about the 63rd percentile of a set of data with 5 or 10 or 20 numbers in it. When you receive your GRE scores in the mail, you will receive a percentile ranking for each of your scores. If you are told that your Verbal score is at the 63rd percentile, that means that your score was higher than the scores of approximately 63% of all GRE test takers (and, therefore, that your score was lower than those of approximately 37% of GRE test takers).

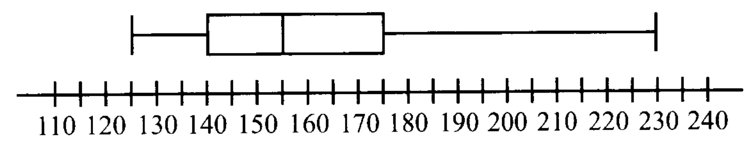
From the definition of percentile, it follows that the median is exactly the same as the 50th percentile. Another term that is often used in analyzing data is ***quartile***. There are three quartiles, Q_1, Q_2, and Q_3, which divide a set of data into four roughly equal groups. Q_1, Q_2, and Q_3 are called the first, second, and third quartiles and are equal to P_{25}, P_{50}, and P_{75}, respectively. So, if M represents the median, then $M = Q_2 = P_{50}$. A measure that is sometimes used to show how spread out the numbers in a set of data are is the ***interquartile range***, which is defined as the difference between the first and third quartiles: $Q_3 - Q_1$.

The interquartile range shows where the middle half of all the data lies. The interquartile range can be graphically illustrated in a diagram called a ***boxplot***. A boxplot extends from the smallest number in the set of data (S) to the largest number in the set of data (L) and has a box representing the interquartile range. The box, which begins and ends at the first and third quartiles, also shows the location of the median (Q_2). The box may be symmetric about the median, but does not need to be, as is illustrated in the two boxplots, below. The upper boxplot shows the distribution of math SAT scores for all students who took the SAT in 2010, while the lower boxplot shows the distribution of math scores for the students at a very selective college.



EXAMPLE 8

Twelve hundred 18-year-old boys were weighed, and their weights, in pounds, are summarized in the following boxplot.



If the 91st percentile of the weights is 200 pounds, approximately how many of the students weigh less than 140 pounds or more than 200 pounds?

Ⓐ 220 Ⓑ 280 Ⓒ 350 Ⓓ 410 Ⓔ 470

SOLUTION.
From the boxplot, we see that the first quartile is 140. So, approximately 25% of the boys weigh less than 140. And since the 91st percentile is 200, approximately 9% of the boys weigh more than 200. So 25% + 9% = 34% of the 1,200 boys fall within the range we are considering.

Finally, 34% of 1,200 = 0.34 × 1200 = 408, or approximately **410 (D)**.

Practice Exercises—Averages

Discrete Quantitative Questions

1. Michael's average (arithmetic mean) on 4 tests is 80. What does he need on his fifth test to raise his average to 84?

   Ⓐ 82
   Ⓑ 84
   Ⓒ 92
   Ⓓ 96
   Ⓔ 100

2. Maryline's average (arithmetic mean) on 4 tests is 80. Assuming she can earn no more than 100 on any test, what is the least she can earn on her fifth test and still have a chance for an 85 average after seven tests?

   Ⓐ 60
   Ⓑ 70
   Ⓒ 75
   Ⓓ 80
   Ⓔ 85

3. Sandrine's average (arithmetic mean) on 4 tests is 80. Which of the following cannot be the number of tests on which she earned exactly 80 points?

   Ⓐ 0
   Ⓑ 1
   Ⓒ 2
   Ⓓ 3
   Ⓔ 4

4. What is the average (arithmetic mean) of the positive integers from 1 to 100, inclusive?

   Ⓐ 49
   Ⓑ 49.5
   Ⓒ 50
   Ⓓ 50.5
   Ⓔ 51

5. If $10a + 10b = 35$, what is the average (arithmetic mean) of a and b?

   [ ]

6. If $x + y = 6$, $y + z = 7$, and $z + x = 9$, what is the average (arithmetic mean) of x, y, and z?

   Ⓐ $\frac{11}{3}$
   Ⓑ $\frac{11}{2}$
   Ⓒ $\frac{22}{3}$
   Ⓓ 11
   Ⓔ 22

7. If the average (arithmetic mean) of 5, 6, 7, and w is 8, what is the value of w?

   Ⓐ 8
   Ⓑ 12
   Ⓒ 14
   Ⓓ 16
   Ⓔ 24

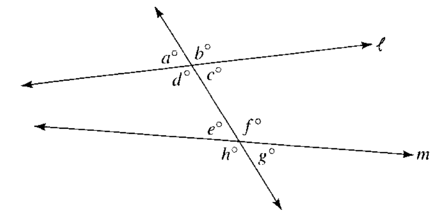
8. What is the average (arithmetic mean) in degrees of the measures of the five angles in a pentagon?

   [ ] degrees

9. If $a + b = 3(c + d)$, which of the following is the average (arithmetic mean) of a, b, c, and d?

   Ⓐ $\frac{c+d}{4}$
   Ⓑ $\frac{3(c+d)}{8}$
   Ⓒ $\frac{c+d}{2}$
   Ⓓ $\frac{3(c+d)}{4}$
   Ⓔ $c + d$

10. In the diagram below, lines ℓ and m are *not* parallel.



If A represents the average (arithmetic mean) of the degree measures of all eight angles, which of the following is true?

   Ⓐ $A = 45$
   Ⓑ $45 < A < 90$
   Ⓒ $A = 90$
   Ⓓ $90 < A < 180$
   Ⓔ $A = 180$

11. What is the average (arithmetic mean) of 2^{10} and 2^{20}?

   Ⓐ 2^{15}
   Ⓑ $2^5 + 2^{10}$
   Ⓒ $2^9 + 2^{19}$
   Ⓓ 2^{29}
   Ⓔ 30

12. Let M be the median and m the mode of the following set of numbers: 10, 70, 20, 40, 70, 90. What is the average (arithmetic mean) of M and m?

   Ⓐ 50
   Ⓑ 55
   Ⓒ 60
   Ⓓ 62.5
   Ⓔ 65

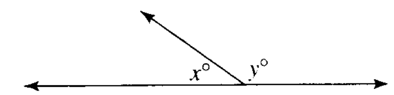
Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

	Quantity A	Quantity B
13.	The average (arithmetic mean) of the measures of the three angles of an equilateral triangle	The average (arithmetic mean) of the measures of the three angles of a right triangle

10 students took a test and the average grade was 80. No one scored exactly 80.

	Quantity A	Quantity B
14.	The number of grades over 80	5



	Quantity A	Quantity B
15.	The average (arithmetic mean) of $2x$ and $2y$	180

There are the same number of boys and girls in a club.
The average weight of the boys is 150 pounds.
The average weight of the girls is 110 pounds.

	Quantity A	Quantity B
16.	The number of boys weighing over 150	The number of girls weighing over 110

The average (arithmetic mean) of 22, 38, x, and y is 15.
$x > 0$

	Quantity A	Quantity B
17.	y	0

	Quantity A	Quantity B
18.	The average (arithmetic mean) of the even numbers between 1 and 11	The average (arithmetic mean) of the odd numbers between 2 and 12

	Quantity A	Quantity B
19.	The average (arithmetic mean) of 17, 217, 417	The average (arithmetic mean) of 0, 17, 217, 417

$y > 0$

	Quantity A	Quantity B
20.	The average (arithmetic mean) of x and y	The average (arithmetic mean) of x, y, and $2y$

ANSWER KEY

1. **E**	5. **1.75**	9. **E**	13. **C**	17. **B**
2. **C**	6. **A**	10. **C**	14. **D**	18. **B**
3. **D**	7. **C**	11. **C**	15. **C**	19. **A**
4. **D**	8. **108**	12. **D**	16. **D**	20. **D**

Answer Explanations

1. **(E)** Use TACTIC E1. For Michael's average on five tests to be an 84, he needs a total of $5 \times 84 = 420$ points. So far, he has earned $4 \times 80 = 320$ points. Therefore, he needs 100 points more.

2. **(C)** Use TACTIC E1. So far, Maryline has earned 320 points. She can survive a low grade on test five if she gets the maximum possible on both the sixth and seventh tests. So, assume she gets two 100s. Then her total for tests 1, 2, 3, 4, 6, and 7 would be 520. For her seven-test average to be 85, she needs a total of $7 \times 85 = 595$ points. Therefore, she needs at least $595 - 520 = 75$ points.

3. **(D)** Since Sandrine's 4-test average is 80, she earned a total of $4 \times 80 = 320$ points. Could Sandrine have earned a total of 320 points with:
0 grades of 80? Easily; for example, 20, 100, 100, 100 or 60, 70, 90, 100.
1 grade of 80? Lots of ways; 80, 40, 100, 100, for instance.
2 grades of 80? Yes; 80, 80, 60, 100.
4 grades of 80? Sure: 80, 80, 80, 80.
3 grades of 80? NO! $80 + 80 + 80 + x = 320 \Rightarrow x = 80$, as well.

4. **(D)** Clearly, the sequence of integers from 1 to 100 has 100 terms, and so by KEY FACT E4, we know that the average of all the numbers is the average of the two middle ones: 50 and 51. The average, therefore, is 50.5.

5. **1.75** Since $10a + 10b = 35$, dividing both sides of the equation by 10, we get that $a + b = 3.5$. Therefore, the average of a and b is $3.5 \div 2 = 1.75$.

6. **(A)** Whenever a question involves three equations, add them:
$$\begin{array}{r} x + y = 6 \\ y + z = 7 \\ + \quad z + x = 9 \\ \hline 2x + 2y + 2z = 22 \end{array}$$
Divide by 2: $x + y + z = 11$
The average of x, y, and z is $\dfrac{x+y+z}{3} = \dfrac{11}{3}$.

7. **(C)** Use TACTIC E1: the sum of the 4 numbers is 4 times their average:
$5 + 6 + 7 + w = 4 \times 8 = 32 \Rightarrow 18 + w = 32 \Rightarrow w = 14$.

8. **108** The average of the measures of the five angles is the sum of their measures divided by 5. The sum is $(5 - 2) \times 180 = 3 \times 180 = 540$ (see Section 11-K). So, the average is $540 \div 5 = 108$.

9. **(E)** Calculate the average:
$$\frac{a+b+c+d}{4} = \frac{3(c+d)+c+d}{4} = \frac{3c+3d+c+d}{4} = \frac{4c+4d}{4} = c + d$$

10. **(C)** Since $a + b + c + d = 360$, and $e + f + g + h = 360$ (see Section 11-I), the sum of the measures of all 8 angles is $360 + 360 = 720$, and their average is $720 \div 8 = 90$.

11. **(C)** The average of 2^{10} and 2^{20} is $\dfrac{2^{10}+2^{20}}{2} = \dfrac{2^{10}}{2} + \dfrac{2^{20}}{2} = 2^9 + 2^{19}$.

12. **(D)** Arrange the numbers in increasing order: 10, 20, 40, 70, 70, 90. M, the median, is the average of the middle two numbers: $\dfrac{40+70}{2} = 55$; the mode, m, is 70, the number that appears most frequently. The average of M and m, therefore, is the average of 55 and 70, which is 62.5.

13. **(C)** In *any* triangle, the sum of the measures of the three angles is 180°, and the average of their measures is $180 \div 3 = 60$.

14. **(D)** From KEY FACT E1, we know only that *at least one grade was above 80.* In fact, there may have been only one (9 grades of 79 and 1 grade of 89, for example). But there could have been five or even nine (for example, 9 grades of 85 and 1 grade of 35).
Alternative solution. The ten students scored exactly 800 points. Ask, "Could they be equal?" Could there be exactly five grades above 80? Sure, five grades of 100 for 500 points and five grades of 60 for 300 points. Must they be equal? No, eight grades of 100 and two grades of 0 also total 800.

15. **(C)** The average of $2x$ and $2y$ is $\dfrac{2x+2y}{2} = x + y$, which equals 180.

16. **(D)** It is possible that no boy weighs over 150 (if every single boy weighs exactly 150); on the other hand, it is possible that almost every boy weighs over 150. The same is true for the girls.

17. **(B)** Use TACTIC E1: $22 + 38 + x + y = 4(15) = 60 \Rightarrow 60 + x + y = 60 \Rightarrow x + y = 0$.
Since it is given that x is positive, y must be negative.

18. **(B)** Don't calculate the averages. Quantity A is the average of 2, 4, 6, 8, and10. Quantity B is the average of 3, 5, 7, 9, and 11. Since each of the five numbers from Quantity A is less than the corresponding number from Quantity B, Quantity A must be less than Quantity B.

19. **(A)** You don't have to calculate the averages. Quantity A is clearly positive, and by KEY FACT E3, adding 0 to the set of numbers being averaged must lower the average.

20. **(D)** Use KEY FACT E3: If $x < y$, then the average of x and y is less than y, and surely less than $2y$. So, $2y$ has to raise the average. On the other hand, if x is much larger than y, then $2y$ would lower the average.

Algebra

For the GRE you need to know only a small portion of the algebra normally taught in a high school elementary algebra course and none of the material taught in an intermediate or advanced algebra course. Sections 11-F, 11-G, and 11-H review only those topics that you absolutely need for the GRE.

11-F. POLYNOMIALS

Even though the terms *monomial*, *binomial*, *trinomial*, and *polynomial* are not used on the GRE, you must be able to work with simple polynomials, and the use of these terms will make it easier for us to discuss the important concepts.

A ***monomial*** is any number or variable or product of numbers and variables. Each of the following is a monomial:

$$3 \quad -4 \quad x \quad y \quad 3x \quad -4xyz \quad 5x^3 \quad 1.5xy^2 \quad a^3b^4$$

The number that appears in front of the variables in a monomial is called the *coefficient*. The coefficient of $5x^3$ is 5. If there is no number, the coefficient is 1 or –1, because x means $1x$ and $-ab^2$ means $-1ab^2$.

On the GRE, you could be asked to evaluate a monomial for specific values of the variables.

EXAMPLE 1

What is the value of $-3a^2b$ when $a = -4$ and $b = 0.5$?

Ⓐ –72 Ⓑ –24 Ⓒ 24 Ⓓ 48 Ⓔ 72

SOLUTION.
Rewrite the expression, replacing the letters a and b with the numbers –4 and 0.5, respectively. Make sure to write each number in parentheses. Then evaluate: $-3(-4)^2(0.5) = -3(16)(0.5) =$ **–24 (B)**.

> **CAUTION**
>
> Be sure you follow PEMDAS (see Section 11-A): handle exponents before the other operations. In Example 1, you *cannot* multiply –4 by –3, get 12, and then square the 12; you must first square –4.

A ***polynomial*** is a monomial or the sum of two or more monomials. Each monomial that makes up the polynomial is called a ***term*** of the polynomial. Each of the following is a polynomial:

$$2x^2 \quad 2x^2 + 3 \quad 3x^2 - 7 \quad x^2 + 5x - 1 \quad a^2b + b^2a \quad x^2 - y^2 \quad w^2 - 2w + 1$$

The first polynomial in the above list is a monomial; the second, third, fifth, and sixth polynomials are called ***binomials***, because each has two terms; the fourth and seventh polynomials are called ***trinomials***, because each has three terms. Two terms are called ***like terms*** if they have exactly the same variables and exponents; they can differ only in their coefficients: $5a^2b$ and $-3a^2b$ are like terms, whereas a^2b and b^2a are not.

The polynomial $3x^2 + 4x + 5x + 2x^2 + x - 7$ has 6 terms, but some of them are like terms and can be combined:

$$3x^2 + 2x^2 = 5x^2 \quad \text{and} \quad 4x + 5x + x = 10x.$$

So, the original polynomial is equivalent to the trinomial $5x^2 + 10x - 7$.

KEY FACT F1

The only terms of a polynomial that can be combined are like terms.

KEY FACT F2

To add two polynomials, put a plus sign between them, erase the parentheses, and combine like terms.

> **TIP**
>
> To add, subtract, multiply, and divide polynomials, use the usual laws of arithmetic. To avoid careless errors, before performing any arithmetic operations, write each polynomial in parentheses.

EXAMPLE 2

What is the sum of $5x^2 + 10x - 7$ and $3x^2 - 4x + 2$?

SOLUTION.

$(5x^2 + 10x - 7) + (3x^2 - 4x + 2)$
$= 5x^2 + 10x - 7 + 3x^2 - 4x + 2$
$= (5x^2 + 3x^2) + (10x - 4x) + (-7 + 2)$
$=$ **$8x^2 + 6x - 5$.**

KEY FACT F3

To subtract two polynomials, change the minus sign between them to a plus sign and change the sign of every term in the second parentheses. Then just use KEY FACT F2 to add them: erase the parentheses and then combine like terms.

> **CAUTION**
>
> Make sure you get the order right in a subtraction problem.

EXAMPLE 3

Subtract $3x^2 - 4x + 2$ from $5x^2 + 10x - 7$.

SOLUTION.
Be careful. Start with the second polynomial and subtract the first:

$(5x^2 + 10x - 7) - (3x^2 - 4x + 2) = (5x^2 + 10x - 7) + (-3x^2 + 4x - 2) = \mathbf{2x^2 + 14x - 9}$.

EXAMPLE 4

What is the average (arithmetic mean) of $5x^2 + 10x - 7$, $3x^2 - 4x + 2$, and $4x^2 + 2$?

SOLUTION.
As in any average problem, add and divide:

$$(5x^2 + 10x - 7) + (3x^2 - 4x + 2) + (4x^2 + 2) = 12x^2 + 6x - 3,$$

and by the distributive law (KEY FACT A21), $\frac{12x^2 + 6x - 3}{3} = \mathbf{4x^2 + 2x - 1}$.

KEY FACT F4

To multiply monomials, first multiply their coefficients, and then multiply their variables (letter by letter), by adding the exponents (see Section 11-A).

EXAMPLE 5

What is the product of $3xy^2z^3$ and $-2x^2y^2$?

SOLUTION.
$(3xy^2z^3)(-2x^2y^2) = 3(-2)(x)(x^2)(y^2)(y^2)(z^3) = \mathbf{-6x^3y^4z^3}$.

All other polynomials are multiplied by using the distributive law.

KEY FACT F5

To multiply a monomial by a polynomial, just multiply each term of the polynomial by the monomial.

EXAMPLE 6

What is the product of $2a$ and $3a^2 - 6ab + b^2$?

SOLUTION.

$2a(3a^2 - 6ab + b^2) = \mathbf{6a^3 - 12a^2b + 2ab^2}$.

On the GRE, the only other polynomials that you could be asked to multiply are two binomials.

KEY FACT F6

To multiply two binomials, use the so-called FOIL method, which is really nothing more than the distributive law: Multiply each term in the first parentheses by each term in the second parentheses and simplify by combining terms, if possible.

$$(2x - 7)(3x + 2) = \underbrace{(2x)(3x)}_{\text{First terms}} + \underbrace{(2x)(2)}_{\text{Outer terms}} + \underbrace{(-7)(3x)}_{\text{Inner terms}} + \underbrace{(-7)(2)}_{\text{Last terms}} =$$

$$6x^2 + 4x - 21x - 14 = 6x^2 - 17x - 14$$

EXAMPLE 7

What is the value of $(x - 2)(x + 3) - (x - 4)(x + 5)$?

[ ]

SOLUTION.
First, multiply both pairs of binomials:

$$(x - 2)(x + 3) = x^2 + 3x - 2x - 6 = x^2 + x - 6$$
$$(x - 4)(x + 5) = x^2 + 5x - 4x - 20 = x^2 + x - 20$$

Now, subtract:

$$(x^2 + x - 6) - (x^2 + x - 20) = x^2 + x - 6 - x^2 - x + 20 = \mathbf{14}.$$

KEY FACT F7

The three most important binomial products on the GRE are these:

- $(x - y)(x + y) = x^2 + xy - yx - y^2 = x^2 - y^2$
- $(x - y)^2 = (x - y)(x - y) = x^2 - xy - yx + y^2 = x^2 - 2xy + y^2$
- $(x + y)^2 = (x + y)(x + y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$

TIP

If you memorize these, you won't have to multiply them out each time you need them.

EXAMPLE 8

If $a - b = 7$ and $a + b = 13$, what is the value of $a^2 - b^2$?

SOLUTION.
In Section 11-G, we will review how to solve such a pair of equations; but even if you know how, *you should not do it here.* You do not need to know the values of a and b to answer this question. The moment you see $a^2 - b^2$, you should think $(a - b)(a + b)$. Then:

$$a^2 - b^2 = (a - b)(a + b) = (7)(13) = \mathbf{91}.$$

EXAMPLE 9

If $x^2 + y^2 = 36$ and $(x + y)^2 = 64$, what is the value of xy?

SOLUTION.

$$64 = (x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 2xy = 36 + 2xy.$$

Therefore, $2xy = 64 - 36 = 28 \Rightarrow xy = \mathbf{14}$.

On the GRE, the only division of polynomials you might have to do is to divide a polynomial by a monomial. You will *not* have to do long division of polynomials.

KEY FACT F8

To divide a polynomial by a monomial, use the distributive law. Then simplify each term by reducing the fraction formed by the coefficients to lowest terms and applying the laws of exponents.

EXAMPLE 10

What is the quotient when $32a^2b + 12ab^3c$ is divided by $8ab$?

SOLUTION.
By the distributive law, $\dfrac{32a^2b + 12ab^3c}{8ab} = \dfrac{32a^2b}{8ab} + \dfrac{12ab^3c}{8ab}$.

Now reduce each fraction: $\mathbf{4a + \frac{3}{2}b^2c}$.

On the GRE, the most important way to use the three formulas in KEY FACT F7 is to recognize them in reverse. In other words, whenever you see $x^2 - y^2$, you should realize that it can be rewritten as $(x - y)(x + y)$. This process, which is the reverse of multiplication, is called ***factoring.***

EXAMPLE 11

Quantity A	Quantity B
The value of $x^2 + 4x + 4$ when $x = 95.9$	The value of $x^2 - 4x + 4$ when $x = 99.5$

SOLUTION.
Obviously, you don't want to plug in 95.9 and 99.5 (remember that the GRE *never* requires you to do tedious arithmetic). Recognize that $x^2 + 4x + 4$ is equal to $(x + 2)^2$ and that $x^2 - 4x + 4$ is equal to $(x - 2)^2$. So, Quantity A is just $(95.9 + 2)^2 = 97.9^2$, whereas Quantity B is $(99.5 - 2)^2 = 97.5^2$. Quantity **A** is greater.

EXAMPLE 12

What is the value of $(1{,}000{,}001)^2 - (999{,}999)^2$?

SOLUTION.
Do not even consider squaring 999,999. You know that there has to be an easier way to do this. In fact, if you stop to think, you can get the right answer in a few seconds. This is just $a^2 - b^2$ where $a = 1{,}000{,}001$ and $b = 999{,}999$, so change it to $(a - b)(a + b)$:

$$(1{,}000{,}001)^2 - (999{,}999)^2 = (1{,}000{,}001 - 999{,}999)(1{,}000{,}001 + 999{,}999) = (2)(2{,}000{,}000) = \mathbf{4{,}000{,}000}.$$

Although the coefficients of any of the terms in a polynomial can be fractions, as in $\frac{2}{3}x^2 - \frac{1}{2}x$, the variable itself cannot be in the denominator. An expression such as $\frac{3 + x}{x^2}$, which does have a variable in the denominator, is called an ***algebraic fraction.*** Fortunately, you should have no trouble with algebraic fractions, since they are handled just like regular fractions. The rules that you reviewed in Section 11-B for adding, subtracting, multiplying, and dividing fractions apply to algebraic fractions, as well.

EXAMPLE 13

What is the sum of the reciprocals of x^2 and y^2?

SOLUTION.

To add $\frac{1}{x^2} + \frac{1}{y^2}$, you need a common denominator, which is x^2y^2.

Multiply the numerator and denominator of $\frac{1}{x^2}$ by y^2 and the numerator and denominator of $\frac{1}{y^2}$ by x^2, and then add:

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{y^2}{x^2y^2} + \frac{x^2}{x^2y^2} = \mathbf{\frac{x^2+y^2}{x^2y^2}}.$$

Often, the way to simplify algebraic fractions is to factor the numerator or the denominator or both. Consider the following example, which is harder than anything you will see on the GRE, but still quite manageable.

EXAMPLE 14

What is the value of $\frac{4x^3 - x}{(2x+1)(6x-3)}$ when $x = 9999$?

SOLUTION.

Don't use FOIL to multiply the denominator. That's going the wrong way. We want to simplify this fraction by factoring everything we can. First factor an x out of the numerator and notice that what's left is the difference of two squares, which can be factored. Then factor out the 3 in the second factor in the denominator:

$$\frac{4x^3 - x}{(2x+1)(6x-3)} = \frac{x(4x^2-1)}{(2x+1)3(2x-1)} = \frac{x\cancel{(2x-1)}\cancel{(2x+1)}}{3\cancel{(2x+1)}\cancel{(2x-1)}} = \frac{x}{3}.$$

So, instead of plugging 9999 into the original expression, plug it into $\frac{x}{3}$: $9999 \div 3 =$ **3333**.

Practice Exercises—Polynomials

Discrete Quantitative Questions

1. What is the value of $\frac{a^2 - b^2}{a - b}$ when $a = 117$ and $b = 118$?

2. If $a^2 - b^2 = 21$ and $a^2 + b^2 = 29$, which of the following could be the value of ab? Indicate *all* possible values.
   - [A] -10
   - [B] $5\sqrt{2}$
   - [C] 10

3. What is the average (arithmetic mean) of $x^2 + 2x - 3$, $3x^2 - 2x - 3$, and $30 - 4x^2$?
   - (A) $\frac{8x^2 + 4x + 24}{3}$
   - (B) $\frac{8x^2 + 24}{3}$
   - (C) $\frac{24 - 4x}{3}$
   - (D) -12
   - (E) 8

4. What is the value of $x^2 + 12x + 36$ when $x = 994$?
   - (A) 11,928
   - (B) 98,836
   - (C) 100,000
   - (D) 988,036
   - (E) 1,000,000

5. If $c^2 + d^2 = 4$ and $(c - d)^2 = 2$, what is the value of cd?
   - (A) 1
   - (B) $\sqrt{2}$
   - (C) 2
   - (D) 3
   - (E) 4

6. What is the value of $(2x + 3)(x + 6) - (2x - 5)(x + 10)$?
   - (A) 32
   - (B) 16
   - (C) 68
   - (D) $4x^2 + 30x + 68$
   - (E) $4x^2 + 30x - 32$

7. If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $ab = c$, what is the average of a and b?
   - (A) 0
   - (B) $\frac{1}{2}$
   - (C) 1
   - (D) $\frac{c}{2}$
   - (E) $\frac{a+b}{2c}$

8. If $x^2 - y^2 = 28$ and $x - y = 8$, what is the average of x and y?
   - (A) 1.75
   - (B) 3.5
   - (C) 7
   - (D) 8
   - (E) 10

9. Which of the following is equal to

$$\left(\frac{1}{a}+a\right)^2 - \left(\frac{1}{a}-a\right)^2?$$

Ⓐ 0
Ⓑ 4
Ⓒ $\frac{1}{a^2} - a^2$
Ⓓ $\frac{2}{a^2} - 2a^2$
Ⓔ $\frac{1}{a^2} - 4 - a^2$

10. If $\left(\frac{1}{a}+a\right)^2 = 100$, what is the value of $\frac{1}{a^2} + a^2$?

Ⓐ 10
Ⓑ 64
Ⓒ 98
Ⓓ 100
Ⓔ 102

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

$n < 0$

	Quantity A	Quantity B
11.	$-2n^2$	$(-2n)^2$

$d < c$

	Quantity A	Quantity B
12.	$(c-d)(c+d)$	$(c-d)(c-d)$

$x = -3$ and $y = 2$

	Quantity A	Quantity B
13.	$-x^2y^3$	0

	Quantity A	Quantity B
14.	$(r+s)(r-s)$	$r(s+r) - s(r+s)$

	Quantity A	Quantity B
15.	$\frac{5x^2-20}{x-2}$	$4x+8$

ANSWER KEY

1. **235**	4. **E**	7. **B**	10. **C**	13. **B**
2. **A, C**	5. **A**	8. **A**	11. **B**	14. **C**
3. **E**	6. **C**	9. **B**	12. **D**	15. **D**

Answer Explanations

1. **235** $\frac{a^2-b^2}{a-b} = \frac{(a-b)(a+b)}{a-b} = a + b = 117 + 118 = 235.$

2. **(A)(C)** Adding the two equations, we get that $2a^2 = 50 \Rightarrow a^2 = 25 \Rightarrow b^2 = 4$. So, $a = 5$ or -5 and $b = 2$ or -2. The only possibilities for their product are 10 and -10. (Only A and C are true.)

3. **(E)** To find the average, take the sum of the three polynomials and then divide by 3. Their sum is $(x^2 + 2x - 3) + (3x^2 - 2x - 3) + (30 - 4x^2) = 24$, and $24 \div 3 = 8$.

4. **(E)** You can avoid messy, time-consuming arithmetic if you recognize that $x^2 + 12x + 36 = (x + 6)^2$. The value is $(994 + 6)^2 = 1000^2 = 1{,}000{,}000$.

5. **(A)** Start by squaring $c - d$: $2 = (c-d)^2 = c^2 - 2cd + d^2 = c^2 + d^2 - 2cd = 4 - 2cd$.
So, $2 = 4 - 2cd \Rightarrow 2cd = 2 \Rightarrow cd = 1$.

6. **(C)** First multiply out both pairs of binomials: $(2x + 3)(x + 6) = 2x^2 + 15x + 18$ and $(2x - 5)(x + 10) = 2x^2 + 15x - 50$.
Now subtract: $(2x^2 + 15x + 18) - (2x^2 + 15x - 50) = 18 - (-50) = 68$.

7. **(B)** $\frac{1}{c} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{a+b}{c} \Rightarrow 1 = a + b \Rightarrow \frac{a+b}{2} = \frac{1}{2}.$

8. **(A)** $x^2 - y^2 = (x-y)(x+y) \Rightarrow 28 = 8(x+y) \Rightarrow x + y = 28 \div 8 = 3.5$.
Finally, the average of x and y is $\frac{x+y}{2} = \frac{3.5}{2} = 1.75$.

9. **(B)** Expand each square: $\left(\frac{1}{a}+a\right)^2 = \frac{1}{a^2} + 2\left(\frac{1}{a}\right)(a) + a^2 = \frac{1}{a^2} + 2 + a^2.$

Similarly, $\left(\frac{1}{a}-a\right)^2 = \frac{1}{a^2} - 2 + a^2.$

Subtract: $\left(\frac{1}{a^2}+2+a^2\right) - \left(\frac{1}{a^2}-2+a^2\right) = 4.$

10. **(C)** $100 = \left(\frac{1}{a}+a\right)^2 = \frac{1}{a^2} + 2 + a^2 \Rightarrow \frac{1}{a^2} + a^2 = 98.$

11. **(B)** Since n is negative, n^2 is positive, and so $-2n^2$ is negative. Therefore, Quantity A is negative, whereas Quantity B is positive.

	Quantity A	Quantity B
12. **(D)** $c > d \Rightarrow c - d$ is positive,		
so divide each side by $c - d$:	$c + d$	$c - d$
Subtract c from each quantity:	d	$-d$

If $d = 0$ the quantities are equal; if $d = 1$, they aren't.

13. **(B)** Quantity A: $-(-3)^2 2^3 = -(9)(8) = -72$.

14. **(C)** Quantity B: $r(s + r) - s(r + s) = rs + r^2 - sr - s^2 = r^2 - s^2$
Quantity A: $(r + s)(r - s) = r^2 - s^2$.

15. **(D)** Quantity A: $\frac{5x^2 - 20}{x - 2} = \frac{5(x^2 - 4)}{x - 2} = \frac{5\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} = 5(x + 2)$.

Quantity B: $4x + 8 = 4(x + 2)$. If $x = -2$, both quantities are 0; for any other value of x the quantities are unequal.

11-G. SOLVING EQUATIONS AND INEQUALITIES

The basic principle that you must adhere to in solving any *equation* is that you can manipulate it in any way, as long as *you do the same thing to both sides*. For example, you may always add the same number to each side; subtract the same number from each side; multiply or divide each side by the same number (except 0); square each side; take the square root of each side (if the quantities are positive); or take the reciprocal of each side. These comments apply to inequalities, as well, except you must be very careful, because some procedures, such as multiplying or dividing by a negative number and taking reciprocals, reverse inequalities (see Section 11-A).

Most of the equations and inequalities that you will have to solve on the GRE have only one variable and no exponents. The following simple six-step method can be used on all of them.

EXAMPLE 1

If $\frac{1}{2}x + 3(x - 2) = 2(x + 1) + 1$, what is the value of x?

SOLUTION.
Follow the steps outlined in the following table.

Step	What to Do	Example 1
1	Get rid of fractions and decimals by multiplying both sides by the Lowest Common Denominator (LCD).	Multiply each term by 2: $x + 6(x - 2) = 4(x + 1) + 2$.
2	Get rid of all parentheses by using the distributive law.	$x + 6x - 12 = 4x + 4 + 2$.
3	Combine like terms on each side.	$7x - 12 = 4x + 6$.
4	By adding or subtracting, get all the variables on one side.	Subtract $4x$ from each side: $3x - 12 = 6$.
5	By adding or subtracting, get all the plain numbers on the other side.	Add 12 to each side: $3x = 18$.
6	Divide both sides by the coefficient of the variable.*	Divide both sides by 3: $x =$ **6**.

*Note: If you start with an inequality and in Step 6 you divide by a negative number, remember to reverse the inequality (see KEY FACT A24).

Example 1 is actually harder than any equation on the GRE, because it required all six steps. On the GRE that never happens. Think of the six steps as a list of questions that must be answered. Ask if each step is necessary. If it isn't, move on to the next one; if it is, do it.

Let's look at Example 2, which does not require all six steps.

EXAMPLE 2

For what real number n is it true that $3(n - 20) = n$?

SOLUTION. Do whichever of the six steps are necessary.

Step	Question	Yes/No	What to Do
1	Are there any fractions or decimals?	No	
2	Are there any parentheses?	Yes	Get rid of them: $3n - 60 = n$.
3	Are there any like terms to combine?	No	
4	Are there variables on both sides?	Yes	Subtract n from each side: $2n - 60 = 0$.
5	Is there a plain number on the same side as the variable?	Yes	Add 60 to each side: $2n = 60$.
6	Does the variable have a coefficient?	Yes	Divide both sides by 2: $n = \mathbf{30}$.

TACTIC G1

Memorize the six steps *in order* and use this method whenever you have to solve this type of equation or inequality.

EXAMPLE 3

Three brothers divided a prize as follows. The oldest received $\frac{2}{5}$ of it, the middle brother received $\frac{1}{3}$ of it, and the youngest received the remaining $120. What was the value of the prize?

SOLUTION.

If x represents the value of the prize, then $\frac{2}{5}x + \frac{1}{3}x + 120 = x$.

Solve this equation using the six-step method.

Step	Question	Yes/No	What to Do
1	Are there any fractions or decimals?	Yes	To get rid of them, multiply by 15. $15\left(\frac{2}{5}x\right) + 15\left(\frac{1}{3}x\right) +$ $15(120) = 15(x)$ $6x + 5x + 1800 = 15x$
2	Are there any parentheses?	No	
3	Are there any like terms to combine?	Yes	Combine them: $11x + 1800 = 15x$.
4	Are there variables on both sides?	Yes	Subtract $11x$ from each side: $1800 = 4x$.
5	Is there a plain number on the same side as the variable?	No	
6	Does the variable have a coefficient?	Yes	Divide both sides by 4: $x = \mathbf{450}$.

Sometimes on the GRE, you are given an equation with several variables and asked to solve for one of them in terms of the others.

TACTIC G2

When you have to solve for one variable in terms of the others, treat all of the others as if they were numbers, and apply the six-step method.

TIP

In applying the six-step method, you shouldn't actually write out the table, as we did in Examples 1-4, since it would be too time consuming. Instead, use the method as a guideline and mentally go through each step, doing whichever ones are required.

EXAMPLE 4

If $a = 3b - c$, what is the value of b in terms of a and c?

SOLUTION.

To solve for b, treat a and c as numbers and use the six-step method with b as the variable.

Step	Question	Yes/No	What to Do
1	Are there any fractions or decimals?	No	
2	Are there any parentheses?	No	
3	Are there any like terms to combine?	No	
4	Are there variables on both sides?	No	Remember: the only variable is b.
5	Is there a plain number on the same side as the variable?	Yes	Remember: we're considering c as a number, and it is on the same side as b, the variable. Add c to both sides: $a + c = 3b$.
6	Does the variable have a coefficient?	Yes	Divide both sides by 3: $b = \mathbf{\frac{a + c}{3}}$.

Sometimes when solving equations, you may see a shortcut. For example, to solve $7(w-3)=42$, it saves time to start by dividing both sides by 7, getting $w-3=6$, rather than by using the distributive law to eliminate the parentheses. Similarly, if you have to solve a proportion such as $\frac{x}{7}=\frac{3}{5}$, it is easier to cross-multiply, getting $5x=21$, than to multiply both sides by 35 to get rid of the fractions (although that's exactly what cross-multiplying accomplishes). Other shortcuts will be illustrated in the problems at the end of the section. If you spot such a shortcut, use it; but if you don't, be assured that the six-step method *always* works.

EXAMPLE 5

If $x-4=11$, what is the value of $x-8$?

(A) −15 (B) −7 (C) −1 (D) 7 (E) 15

SOLUTION.
Going immediately to Step 5, add 4 to each side: $x=15$. But this is *not* the answer. You need the value not of x, but of $x-8$: $15-8=$ **7 (D)**.

As in Example 5, on the GRE you are often asked to solve for something other than the simple variable. In Example 5, you could have been asked for the value of x^2 or $x+4$ or $(x-4)^2$, and so on.

TACTIC

G3

As you read each question on the GRE, on your scrap paper write down whatever you are looking for, and circle it. This way you will always be sure that you are answering the question that is asked.

TIP

Very often, solving the equation is not the quickest way to answer the question. Consider Example 6.

EXAMPLE 6

If $2x-5=98$, what is the value of $2x+5$?

[ ]

SOLUTION.
The first thing you should do is write $2x+5$ on your paper and circle it. The fact that you are asked for the value of something other than x should alert you to look at the question carefully to see if there is a shortcut.

- The best approach here is to observe that $2x+5$ is 10 more than $2x-5$, so the answer is **108** (10 more than 98).
- Next best would be to do only one step of the six-step method, add 5 to both sides: $2x=103$. Now, add 5 to both sides: $2x+5=103+5=108$.
- The *worst* method would be to divide $2x=103$ by 2, get $x=51.5$, and then use that to calculate $2x+5$.

EXAMPLE 7

If w is an integer, and the average (arithmetic mean) of 3, 4, and w is less than 10, what is the greatest possible value of w?

(A) 9 (B) 10 (C) 17 (D) 22 (E) 23

SOLUTION.
Set up the inequality: $\frac{3+4+w}{3}<10$. Do Step 1 (get rid of fractions by multiplying by 3): $3+4+w<30$. Do Step 3 (combine like terms): $7+w<30$. Finally, do Step 5 (subtract 7 from each side): $w<23$. Since w is an integer, the most it can be is **22 (D)**.

The six-step method also works when there are variables in denominators.

EXAMPLE 8

For what value of x is $\frac{4}{x}+\frac{3}{5}=\frac{10}{x}$?

(A) 5 (B) 10 (C) 20 (D) 30 (E) 50

SOLUTION. Multiply each side by the LCD, $5x$:

$$5x\left(\frac{4}{x}\right)+5x\left(\frac{3}{5}\right)=5x\left(\frac{10}{x}\right)\Rightarrow 20+3x=50.$$

Now solve normally: $20+3x=50 \Rightarrow 3x=30$ and so $x=$ **10 (B)**.

EXAMPLE 9

If x is positive, and $y=5x^2+3$, which of the following is an expression for x in terms of y?

(A) $\sqrt{\frac{y}{5}-3}$ (B) $\sqrt{\frac{y-3}{5}}$ (C) $\frac{\sqrt{y-3}}{5}$ (D) $\frac{\sqrt{y}-3}{5}$ (E) $\frac{\sqrt{y}-\sqrt{3}}{5}$

SOLUTION.
The six-step method works when there are no exponents. However, we can treat x^2 as a single variable, and use the method as far as possible:

$$y=5x^2+3 \Rightarrow y-3=5x^2 \Rightarrow \frac{y-3}{5}=x^2.$$

Now take the square root of each side; since x is positive, the only solution is $x=\sqrt{\frac{y-3}{5}}$ **(B)**.

CAUTION

Doing the same thing to each *side* of an equation does *not* mean doing the same thing to each *term* of the equation. Study Examples 10 and 11 carefully.

NOTE

You *cannot* just take the reciprocal of each term; the answer is *not* $a = b + c$. Here are two solutions.

EXAMPLE 10

If $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, what is a in terms of b and c?

SOLUTION 1.
First add the fractions on the right hand side:

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c} = \frac{b+c}{bc}.$$

Now, take the reciprocal of each *side*: $a = \frac{bc}{b+c}$.

SOLUTION 2.
Use the six-step method. Multiply each term by abc, the LCD:

$$\not{a}bc\left(\frac{1}{\not{a}}\right) = a\not{b}c\left(\frac{1}{\not{b}}\right) + ab\not{c}\left(\frac{1}{\not{c}}\right) \Rightarrow bc = ac + ab = a(c + b) \Rightarrow a = \frac{bc}{c+b}.$$

EXAMPLE 11

If $a > 0$ and $a^2 + b^2 = c^2$, what is a in terms of b and c?

SOLUTION. $a^2 + b^2 = c^2 \Rightarrow a^2 = c^2 - b^2$. Be careful: you *cannot* now take the square root of each *term* and write, $a = c - b$. Rather, you must take the square root of each *side*: $a = \sqrt{a^2} = \sqrt{c^2 - b^2}$.

There are a few other types of equations that you could have to solve on the GRE. Fortunately, they are quite easy. You probably will not have to solve a quadratic equation. However, if you do, you will *not* need the quadratic formula, and you will not have to factor a trinomial. Here are two examples.

EXAMPLE 12

If x is a positive number and $x^2 + 64 = 100$, what is the value of x?

Ⓐ 6 Ⓑ 12 Ⓒ 13 Ⓓ 14 Ⓔ 36

SOLUTION. When there is an x^2-term, but no x-term, we just have to take a square root:

$$x^2 + 64 = 100 \Rightarrow x^2 = 36 \Rightarrow x = \sqrt{36} = \mathbf{6}\ \mathbf{(A)}.$$

EXAMPLE 13

What is the largest value of x that satisfies the equation $2x^2 - 3x = 0$?

Ⓐ 0 Ⓑ 1.5 Ⓒ 2 Ⓓ 2.5 Ⓔ 3

SOLUTION.
When an equation has an x^2-term and an x-term but no constant term, the way to solve it is to factor out the x and to use the fact that if the product of two numbers is 0, one of them must be 0 (KEY FACT A3):

$$\begin{aligned} 2x^2 - 3x = 0 &\Rightarrow x(2x - 3) = 0 \\ x = 0 \quad &\text{or} \quad 2x - 3 = 0 \\ x = 0 \quad &\text{or} \quad 2x = 3 \\ x = 0 \quad &\text{or} \quad x = 1.5. \end{aligned}$$

The largest value is **1.5 (B)**.

In another type of equation that occasionally appears on the GRE, the variable is in the exponent. These equations are particularly easy and are basically solved by inspection.

EXAMPLE 14

If $2^{x+3} = 32$, what is the value of 3^{x+2}?

Ⓐ 5 Ⓑ 9 Ⓒ 27 Ⓓ 81 Ⓔ 125

SOLUTION.
How many 2s do you have to multiply together to get 32? If you don't know that it's 5, just multiply and keep track. Count the 2s on your fingers as you say to yourself, "2 times 2 is 4, times 2 is 8, times 2 is 16, times 2 is 32." Then

$$2^{x+3} = 32 = 2^5 \Rightarrow x + 3 = 5 \Rightarrow x = 2.$$

Therefore, $x + 2 = 4$, and $3^{x+2} = 3^4 = 3 \times 3 \times 3 \times 3 = \mathbf{81}$ **(D)**.

Occasionally, both sides of an equation have variables in the exponents. In that case, it is necessary to write both exponentials with the same base.

EXAMPLE 15

If $4^{w+3} = 8^{w-1}$, what is the value of w?

Ⓐ 0 Ⓑ 1 Ⓒ 2 Ⓓ 3 Ⓔ 9

SOLUTION.
Since it is necessary to have the same base on each side of the equation, write $4 = 2^2$ and $8 = 2^3$. Then

$$4^{w+3} = (2^2)^{w+3} = 2^{2(w+3)} = 2^{2w+6} \quad \text{and} \quad 8^{w-1} = (2^3)^{w-1} = 2^{3(w-1)} = 2^{3w-3}.$$

So, $2^{2w+6} = 2^{3w-3} \Rightarrow 2w + 6 = 3w - 3 \Rightarrow w = \mathbf{9}$ **(E)**.

Systems of Linear Equations

The equations $x + y = 10$ and $x - y = 2$ each have lots of solutions (infinitely many, in fact). Some of them are given in the tables below.

$x + y = 10$

x	5	**6**	4	1	1.2	10	20
y	5	**4**	6	9	8.8	0	–10
$x+y$	10	10	10	10	10	10	10

$x - y = 2$

x	5	**6**	2	0	2.5	19	40
y	3	**4**	0	–2	.5	17	38
$x-y$	2	2	2	2	2	2	2

However, only one pair of numbers, $\boldsymbol{x = 6}$ and $\boldsymbol{y = 4}$, satisfy both equations simultaneously: $6 + 4 = 10$ and $6 - 4 = 2$. This then is the only solution of the

system of equations: $\begin{Bmatrix} x + y = 10 \\ x - y = 2 \end{Bmatrix}$.

A system of equations is a set of two or more equations involving two or more variables. To solve such a system, you must find values for each of the variables that will make each equation true. In an algebra course you learn several ways to solve systems of equations. On the GRE, the most useful way to solve them is to add or subtract (usually add) the equations. After demonstrating this method, we will show in Example 19 one other way to handle some systems of equations.

TACTIC

G4

To solve a system of equations, add or subtract them. If there are more than two equations, add them.

EXAMPLE 16

$x + y = 10$
$x - y = 2$

Quantity A	Quantity B
x	y

SOLUTION.
Add the two equations:

$$\begin{aligned} x + y &= 10 \\ + \; x - y &= 2 \\ \hline 2x \quad &= 12 \\ x &= 6 \end{aligned}$$

Replacing x with 6 in $x + y = 10$ yields $y = 4$. So, Quantity **A** is greater.

TIP

On the GRE, most problems involving systems of equations do not require you to solve the system. They usually ask for something other than the values of each variable. Read the questions very carefully, circle what you need, and do no more than is required.

EXAMPLE 17

If $3a + 5b = 10$ and $5a + 3b = 30$, what is the average (arithmetic mean) of a and b?

SOLUTION.
Add the two equations:

$$\begin{aligned} 3a + 5b &= 10 \\ + \; 5a + 3b &= 30 \\ \hline 8a + 8b &= 40 \end{aligned}$$

Divide both sides by 8: $a + b = 5$

The average of a and b is: $\frac{a+b}{2} = \frac{5}{2} = \mathbf{2.5}$

NOTE: It is not only unnecessary to first solve for a and b ($a = 7.5$ and $b = -2.5$.), but, because that procedure is so much more time-consuming, it would be foolish to do so.

EXAMPLE 18

$7a - 3b = 200$
$7a + 3b = 100$

Quantity A	Quantity B
a	b

SOLUTION.
Don't actually solve the system. Add the equations:

$$14a = 300 \Rightarrow 7a = 150.$$

So, replacing $7a$ with 150 in the second equation, we get $150 + 3b = 100$; so $3b$, and hence b, must be negative, whereas a is positive. Therefore, $a > b$, and Quantity **A** is greater.

TIP

Remember TACTIC 5, Chapter 9. On quantitative comparison questions, you don't need to know the value of the quantity in each column; you only need to know which one is greater.

Occasionally on the GRE, it is as easy, or easier, to solve the system by substitution.

TACTIC

G5

If one of the equations in a system of equations consists of a single variable equal to some expression, substitute that expression for the variable in the other equation.

EXAMPLE 19

$$x + y = 10$$
$$y = x - 2$$

Quantity A	Quantity B
x	y

SOLUTION.
Since the second equation states that a single variable (y), is equal to some expression ($x - 2$), substitute that expression for y in the first equation: $x + y = 10$ becomes $x + (x - 2) = 10$. Then, $2x - 2 = 10$, $2x = 12$, and $x = 6$. As always, to find the value of the other variable (y), plug the value of x into one of the two original equations: $y = 6 - 2 = 4$. Quantity **A** is greater.

Practice Exercises — Equations/Inequalities

Discrete Quantitative Questions

1. If $4x + 12 = 36$, what is the value of $x + 3$?
   Ⓐ 3
   Ⓑ 6
   Ⓒ 9
   Ⓓ 12
   Ⓔ 18

2. If $7x + 10 = 44$, what is the value of $7x - 10$?
   Ⓐ $-6\frac{6}{7}$
   Ⓑ $4\frac{6}{7}$
   Ⓒ $14\frac{6}{7}$
   Ⓓ 24
   Ⓔ 34

3. If $4x + 13 = 7 - 2x$, what is the value of x?
   Ⓐ $-\frac{10}{3}$
   Ⓑ -3
   Ⓒ -1
   Ⓓ 1
   Ⓔ $\frac{10}{3}$

4. If $x - 4 = 9$, what is the value of $x^2 - 4$?

   [ ]

5. If $ax - b = c - dx$, what is the value of x in terms of a, b, c, and d?
   Ⓐ $\frac{b+c}{a+d}$
   Ⓑ $\frac{c-b}{a-d}$
   Ⓒ $\frac{b+c-d}{a}$
   Ⓓ $\frac{c-b}{a+d}$
   Ⓔ $\frac{c}{b} - \frac{d}{a}$

6. If $\frac{1}{3}x + \frac{1}{6}x + \frac{1}{9}x = 33$, what is the value of x?
   Ⓐ 3
   Ⓑ 18
   Ⓒ 27
   Ⓓ 54
   Ⓔ 72

7. If $3x - 4 = 11$, what is the value of $(3x - 4)^2$?
   Ⓐ 22
   Ⓑ 36
   Ⓒ 116
   Ⓓ 121
   Ⓔ 256

8. If $64^{12} = 2^{a-3}$, what is the value of a?
   Ⓐ 9
   Ⓑ 15
   Ⓒ 69
   Ⓓ 72
   Ⓔ 75

9. If the average (arithmetic mean) of $3a$ and $4b$ is less than 50, and a is twice b, what is the largest possible integer value of a?
   Ⓐ 9
   Ⓑ 10
   Ⓒ 11
   Ⓓ 19
   Ⓔ 20

10. If $\frac{1}{a-b} = 5$, then $a =$
   Ⓐ $b + 5$
   Ⓑ $b - 5$
   Ⓒ $b + \frac{1}{5}$
   Ⓓ $b - \frac{1}{5}$
   Ⓔ $\frac{1-5b}{5}$

11. If $x = 3a + 7$ and $y = 9a^2$, what is y in terms of x?
   Ⓐ $(x - 7)^2$
   Ⓑ $3(x - 7)^2$
   Ⓒ $\frac{(x-7)^2}{3}$
   Ⓓ $\frac{(x+7)^2}{3}$
   Ⓔ $(x + 7)^2$

12. If $4y - 3x = 5$, what is the smallest integer value of x for which $y > 100$?

   [ ]

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

$a + b = 13$
$a - b = 13$

	Quantity A	Quantity B
13.	b	13

$\frac{2^{a-1}}{2^{b+1}} = 8$

	Quantity A	Quantity B
14.	a	b

$4x^2 = 3x$

	Quantity A	Quantity B
15.	x	1

$a + b = 1$
$b + c = 2$
$c + a = 3$

	Quantity A	Quantity B
16.	The average (arithmetic mean) of a, b, and c	1

$3x - 4y = 5$
$y = 2x$

	Quantity A	Quantity B
17.	x	y

$\frac{x}{2} - 2 > \frac{x}{3}$

	Quantity A	Quantity B
18.	x	12

$3r - 5s = 17$
$2r - 6s = 7$

	Quantity A	Quantity B
19.	The average (arithmetic mean) of r and s	10

$\frac{1}{c} = 1 + \frac{1}{d}$

c and d are positive

	Quantity A	Quantity B
20.	c	d

ANSWER KEY

1. **C**	5. **A**	9. **D**	13. **B**	17. **A**
2. **D**	6. **D**	10. **C**	14. **A**	18. **A**
3. **C**	7. **D**	11. **A**	15. **B**	19. **B**
4. **165**	8. **E**	12. **132**	16. **C**	20. **B**

Answer Explanations

1. **(C)** The easiest method is to recognize that $x + 3$ is $\frac{1}{4}$ of $4x + 12$ and, therefore, equals $\frac{1}{4}$ of 36, which is 9. If you don't see that, solve normally:

   $4x + 12 = 36 \Rightarrow 4x = 24 \Rightarrow x = 6$ and so $x + 3 = 9$.

2. **(D)** Subtracting 20 from each side of $7x + 10 = 44$ gives $7x - 10 = 24$. If you don't see that, subtract 10 from each side, getting $7x = 34$. Then subtract 10 to get $7x - 10 = 24$. The worst alternative is to divide both sides of $7x = 34$ by 7 to get $x = \frac{34}{7}$; then you have to multiply by 7 to get back to 34, and then subtract 10.

3. **(C)** Add $2x$ to each side: $6x + 13 = 7$. Subtract 13 from each side: $6x = -6$. Divide by 6: $x = -1$.

4. **165** $x - 4 = 9 \Rightarrow x = 13 \Rightarrow x^2 = 169$ and so $x^2 - 4 = 165$.

5. **(A)** Treat a, b, c, and d as constants, and use the six-step method to solve for x:

   $ax - b = c - dx \Rightarrow ax - b + dx = c \Rightarrow ax + dx = c + b \Rightarrow x(a + d) = b + c \Rightarrow$

   $x = \frac{b+c}{a+d}$.

6. **(D)** Multiply both sides by 18, the LCD:

$$18\left(\frac{1}{3}x + \frac{1}{6}x + \frac{1}{9}x\right) = 18(33) \Rightarrow 6x + 3x + 2x = 594 \Rightarrow 11x = 594 \Rightarrow x = 54.$$

It's actually easier not to multiply out 18 × 33; leave it in that form, and then divide by 11: $\frac{18 \times \cancel{33}^{3}}{\cancel{11}_{1}} = 3 \times 18 = 54.$

7. **(D)** Be alert. Since you are given the value of $3x - 4$, and want the value of $(3x - 4)^2$, just square both sides: $11^2 = 121$. If you don't see that, you'll waste time solving $3x - 4 = 11(x = 5)$, only to use that value to calculate that $3x - 4$ is equal to 11, which you aready knew.

8. **(E)** $2^{a-3} = 64^{12} = (2^6)^{12} = 2^{72} \Rightarrow a - 3 = 72$, and so $a = 75$.

9. **(D)** Since $a = 2b$, $2a = 4b$. Therefore, the average of $3a$ and $4b$ is the average of $3a$ and $2a$, which is $2.5a$. Therefore, $2.5a < 50 \Rightarrow a < 20$. So the largest *integer* value of a is 19.

10. **(C)** Taking the reciprocal of each side, we get $a - b = \frac{1}{5}$. So $a = b + \frac{1}{5}$.

11. **(A)** $x = 3a + 7 \Rightarrow x - 7 = 3a \Rightarrow a = \frac{x-7}{3}$.

Therefore, $y = 9a^2 = 9\left(\frac{x-7}{3}\right)^2 = 9\frac{(x-7)^2}{3^2} = (x-7)^2$.

12. **132** Solving for y yields $y = \frac{5+3x}{4}$.

Then, since $y > 100$: $\frac{5+3x}{4} > 100 \Rightarrow 5 + 3x > 400 \Rightarrow 3x > 395 \Rightarrow$

$x > 131.666$.

The smallest integer value of x is 132.

13. **(B)** Adding the two equations, we get that $2a = 26$. Therefore, $a = 13$ and $b = 0$.

14. **(A)** Express each side of $\frac{2^{a-1}}{2^{b+1}} = 8$ as a power of 2:

$8 = 2^3$ and $\frac{2^{a-1}}{2^{b+1}} = 2^{(a-1)-(b+1)} = 2^{a-b-2}$.

Therefore, $a - b - 2 = 3 \Rightarrow a = b + 5$, and so a is greater.

15. **(B)** $4x^2 = 3x \Rightarrow 4x^2 - 3x = 0 \Rightarrow x(4x - 3) = 0$.

So,

$x = 0$ or $4x - 3 = 0 \Rightarrow$

$x = 0$ or $4x = 3 \Rightarrow$

$x = 0$ or $x = \frac{3}{4}$.

There are two possible values of x, both of which are less than 1.

16. **(C)** When we add all three equations, we get

$2a + 2b + 2c = 6 \Rightarrow a + b + c = 3$, and so $\frac{a+b+c}{3} = 1$.

17. **(A)** Use substitution. Replace y in the first equation with $2x$:

$3x - 4(2x) = 5 \Rightarrow 3x - 8x = 5 \Rightarrow -5x = 5 \Rightarrow x = -1 \Rightarrow y = -2$.

18. **(A)** Multiply both sides by 6, the LCD:

$6\left(\frac{x}{2} - 2\right) > 6\left(\frac{x}{3}\right) \Rightarrow 3x - 12 > 2x \Rightarrow -12 > -x \Rightarrow x > 12$.

19. **(B)** The first thing to try is to add the equations. That yields $5r - 11s = 24$, which does not appear to be useful. So now try to subtract the equations. That yields $r + s = 10$.

So the average of r and s is $\frac{r+s}{2} = \frac{10}{2} = 5$.

20. **(B)** Multiply both sides of the given equation by cd, the LCD of the fractions:

$cd\left(\frac{1}{c}\right) = cd\left(1 + \frac{1}{d}\right) \Rightarrow d = cd + c = c(d + 1) \Rightarrow c = \frac{d}{d+1}$.

Since d is positive, $d + 1 > 1$, and so $\frac{d}{d+1} < d$.

So $c < d$.

11-H. WORD PROBLEMS

On a typical GRE you will see several word problems, covering almost every math topic for which you are responsible. In this chapter you have already seen word problems on consecutive integers in Section A; fractions and percents in Sections B and C; ratios and rates in Section D; and averages in Section E. Later in this chapter you will see word problems involving probability, circles, triangles, and other geometric figures. A few of these problems can be solved with just arithmetic, but most of them require basic algebra.

To solve word problems algebraically, you must treat algebra as a foreign language and learn to translate "word for word" from English into algebra, just as you would from English into French or Spanish or any other language. When translating into algebra, we use some letter (often x) to represent the unknown quantity we are trying to determine. It is this translation process that causes difficulty for some students. Once translated, solving is easy using the techniques we have already reviewed. Consider the following pairs of typical GRE questions. The first ones in each pair (1A and 2A) would be considered easy, whereas the second ones (1B and 2B) would be considered harder.

EXAMPLE 1A

What is 4% of 4% of 40,000?

EXAMPLE 1B

In a lottery, 4% of the tickets printed can be redeemed for prizes, and 4% of those tickets have values in excess of $100. If the state prints 40,000 tickets, how many of them can be redeemed for more than $100?

EXAMPLE 2A

If $x + 7 = 2(x - 8)$, what is the value of x?

EXAMPLE 2B

In 7 years Erin will be twice as old as she was 8 years ago. How old is Erin now?

Once you translate the words into arithmetic expressions or algebraic equations, Examples 1A and 1B and 2A and 2B are identical. The problem that many students have is doing the translation. It really isn't very difficult, and we'll show you how. First, though, look over the following English to algebra "dictionary."

English Words	Mathematical Meaning	Symbol
Is, was, will be, had, has, will have, is equal to, is the same as	Equals	=
Plus, more than, sum, increased by, added to, exceeds, received, got, older than, farther than, greater than	Addition	+
Minus, fewer, less than, difference, decreased by, subtracted from, younger than, gave, lost	Subtraction	−
Times, of, product, multiplied by	Multiplication	×
Divided by, quotient, per, for	Division	÷, $\frac{a}{b}$
More than, greater than	Inequality	>
At least	Inequality	≥
Fewer than, less than	Inequality	<
At most	Inequality	≤
What, how many, etc.	Unknown quantity	x (or some other variable)

Let's use our dictionary to translate some phrases and sentences.

1. The sum of 5 and some number is 13. $5 + x = 13$
2. John was 2 years younger than Sam. $J = S - 2$
3. Bill has at most $100. $B \leq 100$
4. The product of 2 and a number exceeds that number by 5 (is 5 more than). $2N = N + 5$

In translating statements, you first must decide what quantity the variable will represent. Often it's obvious. Other times there is more than one possibility.

Let's translate and solve the two questions from the beginning of this section, and then we'll look at a few new ones.

TIP

In all word problems on the GRE, remember to write down and circle what you are looking for. Don't answer the wrong question!

EXAMPLE 1B

In a lottery, 4% of the tickets printed can be redeemed for prizes, and 4% of those tickets have values in excess of $100. If the state prints 40,000 tickets, how many of them can be redeemed for more than $100?

[ ]

SOLUTION.
Let x = the number of tickets worth more than $100. Then

$$x = 4\% \text{ of } 4\% \text{ of } 40{,}000 = .04 \times .04 \times 40{,}000 = \mathbf{64},$$

which is also the solution to Example 1a.

EXAMPLE 2B

In 7 years Erin will be twice as old as she was 8 years ago. How old is Erin now?

[ ]

SOLUTION.
Let x = Erin's age now. Then 8 years ago she was $x - 8$, and 7 years from now she will be $x + 7$. So,

$$x + 7 = 2(x - 8) \Rightarrow x + 7 = 2x - 16 \Rightarrow 7 = x - 16 \Rightarrow x = \mathbf{23},$$

which is also the solution to Example 2a.

TIP

In problems involving ages, remember that "years ago" means you need to subtract, and "years from now" means you need to add.

Most algebraic word problems on the GRE are not too difficult, and if you can do the algebra, that's usually the best way. But if, after studying this section, you still get stuck on a question during the test, don't despair. Use the tactics that you learned in Chapter 8, especially TACTIC 1—backsolving.

Age Problems

EXAMPLE 3

In 1980, Judy was 3 times as old as Adam, but in 1984 she was only twice as old as he was. How old was Adam in 1990?

Ⓐ 4 Ⓑ 8 Ⓒ 12 Ⓓ 14 Ⓔ 16

SOLUTION.
Let x be Adam's age in 1980 and fill in the table below.

TIP

It is often very useful to organize the data from a word problem in a table.

Year	Judy	Adam
1980	$3x$	x
1984	$3x + 4$	$x + 4$

Now translate: Judy's age in 1984 was twice Adam's age in 1984:

$$3x + 4 = 2(x + 4) = 2x + 8$$

$$3x + 4 = 2x + 8 \Rightarrow x + 4 = 8, \text{ and so } x = 4.$$

So, Adam was 4 in 1980. However, 4 is *not* the answer to this question. Did you remember to circle what you're looking for? The question *could have* asked for Adam's age in 1980 (Choice A) or 1984 (Choice B) or Judy's age in any year whatsoever (Choice C is 1980 and Choice E is 1984); but it didn't. It asked for *Adam's age in 1990*. Since he was 4 in 1980, then 10 years later, in 1990, he was **14 (D)**.

Distance Problems

Distance problems all depend on three variations of the same formula:

$$\textbf{distance} = \textbf{rate} \times \textbf{time} \qquad \textbf{rate} = \frac{\textbf{distance}}{\textbf{time}} \qquad \textbf{time} = \frac{\textbf{distance}}{\textbf{rate}}$$

These are usually abbreviated, $\boldsymbol{d = rt}$, $\boldsymbol{r = \frac{d}{t}}$, and $\boldsymbol{t = \frac{d}{r}}$.

EXAMPLE 4

How much longer, in *seconds*, is required to drive 1 mile at 40 miles per hour than at 60 miles per hour?

[ ] seconds

SOLUTION.
The time to drive 1 mile at 40 miles per hour is given by

$$t = \frac{1}{40}\text{ hour} = \frac{1}{\cancel{40}_2} \times \overset{3}{\cancel{60}}\text{ minutes} = 1\frac{1}{2}\text{ minutes.}$$

The time to drive 1 mile at 60 miles per hour is given by $t = \frac{1}{60}$ hour = 1 minute.

The difference is $\frac{1}{2}$ minute = **30** seconds.

Note that this solution used the time formula given, but required only arithmetic, not algebra. Example 5 requires an algebraic solution.

EXAMPLE 5

Avi drove from his home to college at 60 miles per hour. Returning over the same route, there was a lot of traffic, and he was only able to drive at 40 miles per hour. If the return trip took 1 hour longer, how many miles did he drive each way?

Ⓐ 2 Ⓑ 3 Ⓒ 5 Ⓓ 120 Ⓔ 240

SOLUTION.
Let x = the number of hours Avi took going to college and make a table.

	rate	time	distance
Going	60	x	$60x$
Returning	40	$x + 1$	$40(x + 1)$

Since he drove the same distance going and returning,

$$60x = 40(x + 1) \Rightarrow 60x = 40x + 40 \Rightarrow 20x = 40, \text{ and so } x = 2.$$

Now be sure to answer the correct question. When $x = 2$, Choices A, B, and C are the time in hours that it took going, returning, and round-trip; Choices D and E are the distances each way and round-trip. You could have been asked for any of the five. If you circled what you're looking for, you won't make a careless mistake. Avi drove **120** miles each way, and so the correct answer is **D**.

The d in $d = rt$ stands for "distance," but it could really be any type of work that is performed at a certain rate, r, for a certain amount of time, t. Example 5 need not be about distance. Instead of driving 120 miles at 60 miles per hour for 2 hours, Avi could have read 120 pages at a rate of 60 pages per hour for 2 hours; or planted 120 flowers at the rate of 60 flowers per hour for 2 hours; or typed 120 words at a rate of 60 words per minute for 2 minutes.

Examples 6 and 7 illustrate two additional word problems of the type that you might find on the GRE.

EXAMPLE 6

Lindsay is trying to collect all the cards in a special commemorative set of baseball cards. She currently has exactly $\frac{1}{4}$ of the cards in that set. When she gets 10 more cards, she will then have $\frac{1}{3}$ of the cards. How many cards are in the set?

Ⓐ 30 Ⓑ 60 Ⓒ 120 Ⓓ 180 Ⓔ 240

SOLUTION.
Let x be the number cards in the set. First, translate this problem from English into algebra: $\frac{1}{4}x + 10 = \frac{1}{3}x$. Now, use the six-step method of Section 11-G to solve the equation. Multiply by 12 to get, $3x + 120 = 4x$, and then subtract $3x$ from each side: x = **120** (**C**).

EXAMPLE 7

Jen, Ken, and Len have a total of $390. Jen has 5 times as much as Len, and Ken has $\frac{3}{4}$ as much as Jen. How much money does Ken have?

Ⓐ $40 Ⓑ $78 Ⓒ $150 Ⓓ $195 Ⓔ $200

Suppose, for example, that in this problem you let x represent the amount of money that Ken has. Then since Ken has $\frac{3}{4}$ as much as Jen, Jen has $\frac{4}{3}$ as much as Ken: $\frac{4}{3}x$; and Jen would have $\frac{1}{5}$ of that: $\left(\frac{1}{5}\right)\left(\frac{4}{3}x\right)$. It is much easier here to let x represent the amount of money Len has.

TIP
You often have a choice as to what to let the variable represent. Don't necessarily let it represent what you're looking for; rather, choose what will make the problem easiest to solve.

SOLUTION.
Let x represent the amount of money Len has. Then $5x$ is the amount that Jen has, and $\frac{3}{4}(5x)$ is the amount that Ken has. Since the total amount of money is \$390, $x + 5x + \frac{15}{4}x = 390$.

Multiply by 4 to get rid of the fraction: $4x + 20x + 15x = 1560$.
Combine like terms and then divide: $39x = 1560 \Rightarrow x = 40$.

So Len has \$40, Jen has 5×40 = \$200, and Ken has $\frac{3}{4}(200)$ = **\$150** (**C**).

Practice Exercises — Word Problems

Discrete Quantitative Questions

1. Howard has three times as much money as Ronald. If Howard gives Ronald \$50, Ronald will then have three times as much money as Howard. How much money, in dollars, do the two of them have together?

   [ ] dollars

2. In the afternoon, Beth read 100 pages at the rate of 60 pages per hour; in the evening, when she was tired, she read another 100 pages at the rate of 40 pages per hour. What was her average rate of reading for the day?
   - Ⓐ 45
   - Ⓑ 48
   - Ⓒ 50
   - Ⓓ 52
   - Ⓔ 55

3. If the sum of five consecutive integers is S, what is the largest of those integers in terms of S?
   - Ⓐ $\frac{S-10}{5}$
   - Ⓑ $\frac{S+4}{4}$
   - Ⓒ $\frac{S+5}{4}$
   - Ⓓ $\frac{S-5}{2}$
   - Ⓔ $\frac{S+10}{5}$

4. As a fund-raiser, the school band was selling two types of candy: lollipops for 40 cents each and chocolate bars for 75 cents each. On Monday, they sold 150 candies and raised 74 dollars. How many lollipops did they sell?
   - Ⓐ 75
   - Ⓑ 90
   - Ⓒ 96
   - Ⓓ 110
   - Ⓔ 120

5. A jar contains only red, white, and blue marbles. The number of red marbles is $\frac{4}{5}$ the number of white ones, and the number of white ones is $\frac{3}{4}$ the number of blue ones. If there are 470 marbles in all, how many of them are blue?
   - Ⓐ 120
   - Ⓑ 135
   - Ⓒ 150
   - Ⓓ 184
   - Ⓔ 200

6. The number of shells in Judy's collection is 80% of the number in Justin's collection. If Justin has 80 more shells than Judy, how many shells do they have altogether?

   [ ] shells

7. What is the greater of two numbers whose product is 900, if the sum of the two numbers exceeds their difference by 30?
   - Ⓐ 15
   - Ⓑ 60
   - Ⓒ 75
   - Ⓓ 90
   - Ⓔ 100

8. On a certain project the only grades awarded were 80 and 100. If 10 students completed the project and the average of their grades was 94, how many earned 100?
   - Ⓐ 2
   - Ⓑ 3
   - Ⓒ 5
   - Ⓓ 7
   - Ⓔ 8

9. If $\frac{1}{2}x$ years ago Adam was 12, and $\frac{1}{2}x$ years from now he will be $2x$ years old, how old will he be $3x$ years from now?
   - Ⓐ 18
   - Ⓑ 24
   - Ⓒ 30
   - Ⓓ 54
   - Ⓔ It cannot be determined from the information given.

10. Since 1950, when Barry was discharged from the army, he has gained 2 pounds every year. In 1980 he was 40% heavier than in 1950. What percent of his 1995 weight was his 1980 weight?
    - Ⓐ 80
    - Ⓑ 85
    - Ⓒ 87.5
    - Ⓓ 90
    - Ⓔ 95

Quantitative Comparison Questions

- Ⓐ Quantity A is greater.
- Ⓑ Quantity B is greater.
- Ⓒ Quantities A and B are equal.
- Ⓓ It is impossible to determine which quantity is greater.

Lindsay is twice as old as she was 10 years ago.
Kimberly is half as old as she will be in 10 years.

	Quantity A	Quantity B
11.	Lindsay's age now	Kimberly's age now

Boris spent $\frac{1}{4}$ of his take-home pay on Saturday and $\frac{1}{3}$ of what was left on Sunday. The rest he put in his savings account.

	Quantity A	Quantity B
12.	The amount of his take-home pay that he spent	The amount of his take-home pay that he saved

In 8 years, Tiffany will be 3 times as old as she is now.

	Quantity A	Quantity B
13.	The number of years until Tiffany will be 6 times as old as she is now	16

Rachel put exactly 50 cents worth of postage on an envelope using only 4-cent stamps and 7-cent stamps.

	Quantity A	Quantity B
14.	The number of 4-cent stamps she used	The number of 7-cent stamps she used

Car A and Car B leave from the same spot at the same time.
Car A travels due north at 40 mph.
Car B travels due east at 30 mph.

	Quantity A	Quantity B
15.	Distance from Car A to Car B 9 hours after they left	450 miles

ANSWER KEY

1. **100**	4. **D**	7. **B**	10. **C**	13. **A**
2. **B**	5. **E**	8. **D**	11. **A**	14. **D**
3. **E**	6. **720**	9. **D**	12. **C**	15. **C**

Answer Explanations

1. **100**

	Ronald	Howard
At the beginning	x	$3x$
After the gift	$x + 50$	$3x - 50$

After the gift, Ronald will have 3 times as much money as Howard:
$x + 50 = 3(3x - 50) \Rightarrow x + 50 = 9x - 150 \Rightarrow 8x = 200$, and so $x = 25$.
So Ronald has \$25 and Howard has \$75, for a total of \$100.

2. **(B)** Beth's average rate of reading is determined by dividing the total number of pages she read (200) by the total amount of time she spent reading. In the afternoon she read for $\frac{100}{60} = \frac{5}{3}$ hours, and in the evening for $\frac{100}{40} = \frac{5}{2}$ hours, for a total time of $\frac{5}{3} + \frac{5}{2} = \frac{10}{6} + \frac{15}{6} = \frac{25}{6}$ hours. So, her average rate was

$200 \div \frac{25}{6} = 200 \times \frac{6}{25} = 48$ pages per hour.

3. **(E)** Let the 5 consecutive integers be $n, n + 1, n + 2, n + 3, n + 4$. Then,
$S = n + n + 1 + n + 2 + n + 3 + n + 4 = 5n + 10 \Rightarrow 5n = S - 10 \Rightarrow n = \frac{S-10}{5}$.
Choice A, therefore, is the *smallest* of the integers; the *largest* is
$n + 4 = \frac{S-10}{5} + 4 = \frac{S-10}{5} + \frac{20}{5} = \frac{S+10}{5}$.

4. **(D)** Let x represent the number of chocolate bars sold; then $150 - x$ is the number of lollipops sold. We must use the same units, so we could write 75 cents as .75 dollars or 74 dollars as 7400 cents. Let's avoid the decimals: x chocolates sold for $75x$ cents and $(150 - x)$ lollipops sold for $40(150 - x)$ cents. So,
$7400 = 75x + 40(150 - x) = 75x + 6000 - 40x = 6000 + 35x \Rightarrow$
$1400 = 35x \Rightarrow x = 40$ and $150 - 40 = 110$.

5. **(E)** If b is the number of blue marbles, then there are $\frac{3}{4}b$ white ones, and $\frac{4}{5}\left(\frac{3}{4}b\right) = \frac{3}{5}b$ red ones.

Therefore, $470 = b + \frac{3}{4}b + \frac{3}{5}b = b\left(1 + \frac{3}{4} + \frac{3}{5}\right) = \frac{47}{20}b$.

So, $b = 470 \div \frac{47}{20} = \overset{10}{\cancel{470}} \times \frac{20}{\cancel{47}_1} = 200$.

6. **720** If x is the number of shells in Justin's collection, then Judy has $.80x$. Since Justin has 80 more shells than Judy,
$x = .80x + 80 \Rightarrow .20x = 80 \Rightarrow x = 80 \div .20 = 400$.
So Justin has 400 and Judy has 320: a total of 720.

7. **(B)** If x represents the greater and y the smaller of the two numbers, then $(x + y) = 30 + (x - y) \Rightarrow y = 30 - y \Rightarrow 2y = 30$, and so $y = 15$. Since $xy = 900$, $x = 900 \div 15 = 60$.

8. **(D)** If x represents the number of students earning 100, then $10 - x$ is the number of students earning 80. So
$94 = \frac{100x + 80(10-x)}{10} \Rightarrow 94 = \frac{100x + 800 - 80x}{10} = \frac{20x + 800}{10} \Rightarrow$
$94 \times 10 = 940 = 20x + 800 \Rightarrow 140 = 20x$, and $x = 7$.

9. **(D)** Since $\frac{1}{2}x$ years ago, Adam was 12, he is now $12 + \frac{1}{2}x$. So $\frac{1}{2}x$ years from now, he will be $12 + \frac{1}{2}x + \frac{1}{2}x = 12 + x$. But, we are told that at that time he will be $2x$ years old. So, $12 + x = 2x \Rightarrow x = 12$.
Thus, he is now $12 + 6 = 18$, and $3x$ or 36 years from now he will be $18 + 36 = 54$.

10. **(C)** Let x be Barry's weight in 1950. By 1980, he had gained 60 pounds (2 pounds per year for 30 years) and was 40% heavier: $60 = .40x \Rightarrow$ $x = 60 \div .4 = 150$. So in 1980, he weighed 210. Fifteen years later, in 1995, he weighed 240: $\frac{210}{240} = \frac{7}{8} = 87.5\%$.

11. **(A)** You can do the simple algebra, but you might realize that if in the past 10 years Lindsay's age doubled, she was 10 and is now 20. Similarly, Kimberly is now 10 and in 10 years will be 20.
Here is the algebra: if x represents Lindsay's age now,
$x = 2(x - 10) \Rightarrow x = 2x - 20 \Rightarrow x = 20$.
Similarly, Kimberly is now 10 and will be 20 in 10 years.

12. **(C)** Let x represent the amount of Boris's take-home pay. On Saturday, he spent $\frac{1}{4}x$ and still had $\frac{3}{4}x$; but on Sunday, he spent $\frac{1}{3}$ of that:

$\frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{4}x\right) = \frac{1}{4}x$. Therefore, he spent $\frac{1}{4}$ of his take-home pay each day.

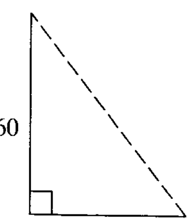
So, he spent $\frac{1}{2}$ of his pay and saved $\frac{1}{2}$ of his pay.

13. **(A)** If x represents Tiffany's age now, then in 8 years she will be $x + 8$, and so $x + 8 = 3x \Rightarrow 8 = 2x \Rightarrow x = 4$.

Tiffany will be 6 times as old 20 years from now, when she will be 24.

14. **(D)** If x and y represent the number of 4-cent stamps and 7-cent stamps that Rachel used, respectively, then $4x + 7y = 50$. This equation has infinitely many solutions but only 2 in which x and y are both positive integers: $y = 2$ and $x = 9$ or $y = 6$ and $x = 2$.

15. **(C)** Draw a diagram. In 9 hours Car A drove 360 miles north and Car B drove 270 miles east. These are the legs of a right triangle, whose hypotenuse is the distance between them. Use the Pythagorean theorem if you don't recognize that this is just a $3x$–$4x$–$5x$ right triangle: the legs are 90×3 and 90×4, and the hypotenuse is $90 \times 5 = 450$.



Geometry

Although about 30% of the math questions on the GRE have to do with geometry, there are only a relatively small number of facts you need to know — far less than you would learn in a geometry course — and, of course, there are no proofs. In the next six sections we will review all of the geometry that you need to know to do well on the GRE. We will present the material exactly as it appears on the GRE, using the same vocabulary and notation, which might be slightly different from the terminology you learned in your high school math classes. The numerous examples in the next six sections will show you exactly how these topics are treated on the GRE.

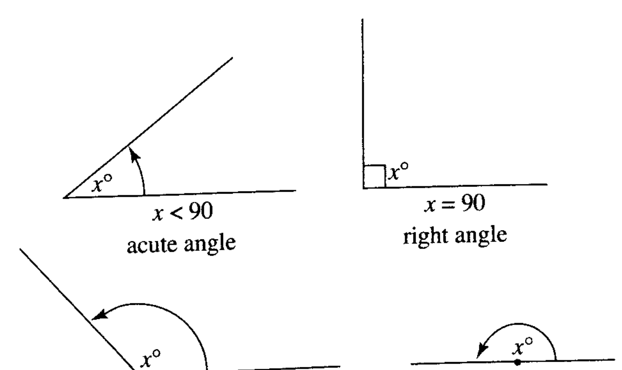
11-I. LINES AND ANGLES

An ***angle*** is formed by the intersection of two line segments, rays, or lines. The point of intersection is called the ***vertex***. On the GRE, angles are always measured in degrees.

KEY FACT I1

Angles are classified according to their degree measures.

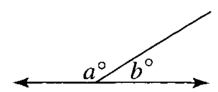
- **An acute angle measures less than 90°.**
- **A right angle measures 90°.**
- **An obtuse angle measures more than 90° but less than 180°.**
- **A straight angle measures 180°.**



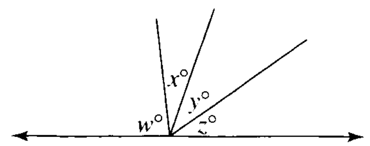
NOTE: The small square in the second angle in the figure above is *always* used to mean that the angle is a right angle. On the GRE, if an angle has a square in it, it must measure exactly 90°, *whether or not you think that the figure has been drawn to scale.*

KEY FACT I2

If two or more angles form a straight angle, the sum of their measures is 180°.



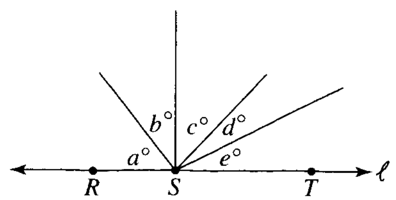
$a + b = 180$



$w + x + y + z = 180$

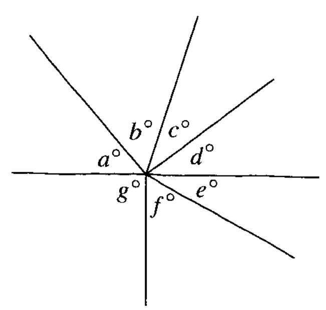
EXAMPLE 1

In the figure below, *R*, *S*, and *T* are all on line ℓ. What is the average of *a*, *b*, *c*, *d*, and *e*?

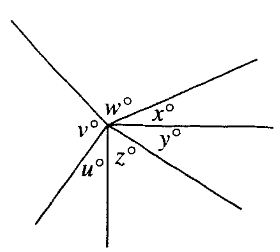


SOLUTION.
Since $\angle RST$ is a straight angle, by KEY FACT I2, the sum of *a*, *b*, *c*, *d*, and *e* is 180, and so their average is $\frac{180}{5}$ = **36.**

In the figure below, since $a + b + c + d = 180$ and $e + f + g = 180$, $a + b + c + d + e + f + g = 180 + 180 = 360$.



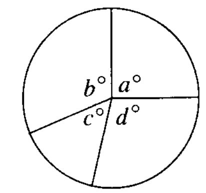
It is also true that $u + v + w + x + y + z = 360$, even though none of the angles forms a straight angle.



KEY FACT I3

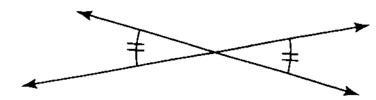
The sum of all the measures of all the angles around a point is 360°.

NOTE: This fact is particularly important when the point is the center of a circle, as we shall see in Section 11-L.



$a + b + c + d = 360$

When two lines intersect, four angles are formed. The two angles in each pair of opposite angles are called ***vertical angles.***



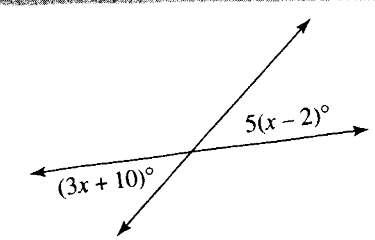
KEY FACT I4

Vertical angles have equal measures.

EXAMPLE 2

In the figure at the right, what is the value of *x*?

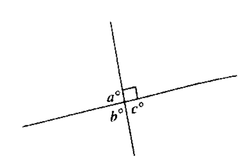
Ⓐ 6 Ⓑ 8 Ⓒ 10 Ⓓ 20 Ⓔ 40



SOLUTION.
Since the measures of vertical angles are equal, $3x + 10 = 5(x - 2) \Rightarrow 3x + 10 = 5x - 10 \Rightarrow 3x + 20 = 5x \Rightarrow 20 = 2x \Rightarrow x =$ **10 (C).**

KEY FACT I5

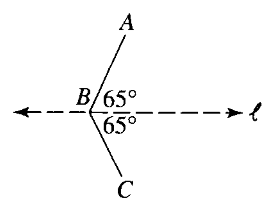
If one of the angles formed by the intersection of two lines (or line segments) is a right angle, then all four angles are right angles.

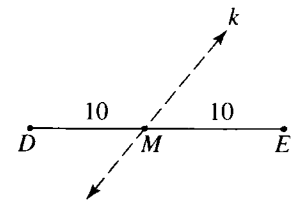


$a = b = c = 90$

Two lines that intersect to form right angles are called ***perpendicular.***

In the figures below, line ℓ divides $\angle ABC$ into two equal parts, and line k divides line segment DE into two equal parts. Line ℓ is said to ***bisect*** the angle, and line k ***bisects*** the line segment. Point M is called the ***midpoint*** of segment DE.

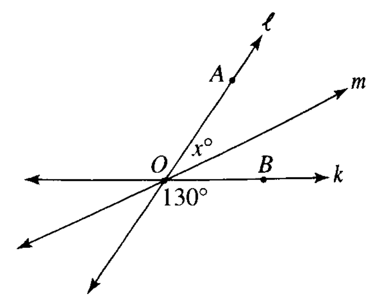




EXAMPLE 3

In the figure at the right, lines k, ℓ, and m intersect at O. If line m bisects $\angle AOB$, what is the value of x?

Ⓐ 25 Ⓑ 35 Ⓒ 45 Ⓓ 50 Ⓔ 60

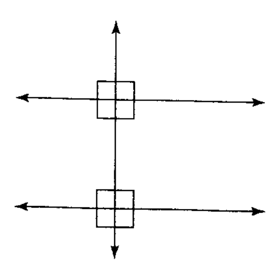


SOLUTION.
$m\angle AOB + 130 = 180 \Rightarrow m\angle AOB = 50$; and since m bisects $\angle AOB$, $x =$ **25 (A)**.

Two lines that never intersect are said to be parallel. Consequently, parallel lines form no angles. However, if a third line, called a ***transversal***, intersects a pair of parallel lines, eight angles are formed, and the relationships among these angles are very important.

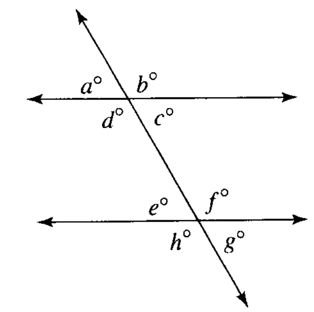
KEY FACT I6

If a pair of parallel lines is cut by a transversal that is perpendicular to the parallel lines, all eight angles are right angles.



KEY FACT I7

If a pair of parallel lines is cut by a transversal that is not perpendicular to the parallel lines,



- **Four of the angles are acute and four are obtuse;**
- **The four acute angles are equal: $a = c = e = g$;**
- **The four obtuse angles are equal: $b = d = f = h$;**
- **The sum of any acute angle and any obtuse angle is 180°: for example, $d + e = 180$, $c + f = 180$, $b + g = 180$, ...**

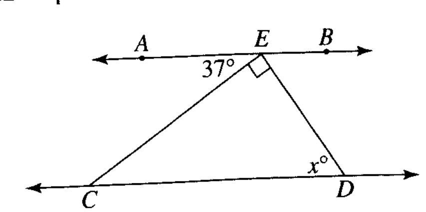
KEY FACT I8

If a pair of lines that are not parallel is cut by a transversal, *none* of the properties listed in KEY FACT I7 is true.

You must know KEY FACT I7 — virtually every GRE has at least one question based on it. However, you do *not* need to know the special terms you learned in high school for these pairs of angles; those terms are not used on the GRE.

EXAMPLE 4

In the figure below, AB is parallel to CD. What is the value of x?



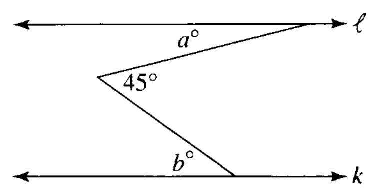
SOLUTION.
Let y be the measure of $\angle BED$. Then by KEY FACT I2:

$$37 + 90 + y = 180 \Rightarrow 127 + y = 180 \Rightarrow y = 53.$$

Since AB is parallel to CD, by KEY FACT I7, $x = y \Rightarrow x =$ **53**.

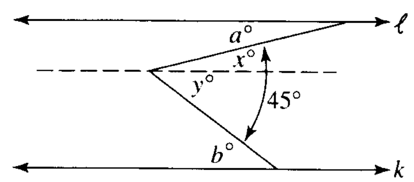
EXAMPLE 5

In the figure below, lines ℓ and k are parallel. What is the value of $a + b$?

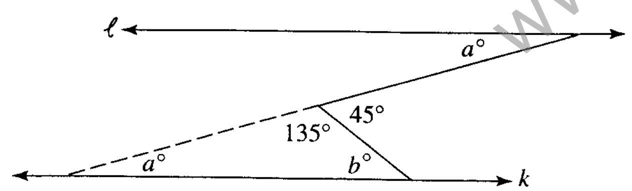


Ⓐ 45 Ⓑ 60 Ⓒ 75 Ⓓ 90 Ⓔ 135

SOLUTION.
It is impossible to determine the value of either a or b. We can, however, find the value of $a + b$. We draw a line through the vertex of the angle parallel to ℓ and k. Then, looking at the top two lines, we see that $a = x$, and looking at the bottom two lines, we see that $b = y$. So, $a + b = x + y =$ **45 (A)**.



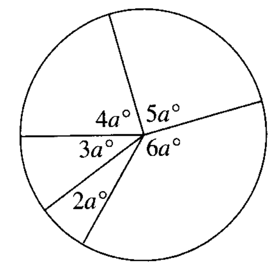
Alternative solution. Draw a different line and use a Key Fact from Section 11-J on triangles. Extend one of the line segments to form a triangle. Since ℓ and k are parallel, the measure of the third angle in the triangle equals a. Now, use the fact that the sum of the measures of the three angles in a triangle is 180° or, even easier, that the given 45° angle is an external angle of the triangle, and so is equal to the sum of a and b.



Practice Exercises — Lines and Angles

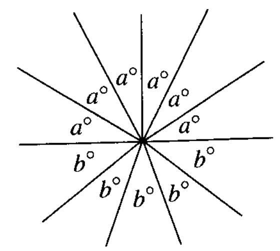
Discrete Quantitative Questions

1. In the figure below, what is the average (arithmetic mean) of the measures of the five angles?



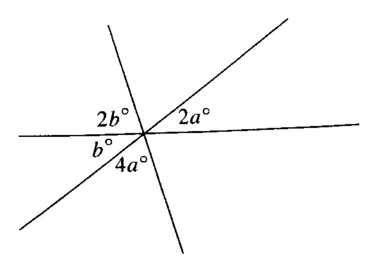
Ⓐ 36
Ⓑ 45
Ⓒ 60
Ⓓ 72
Ⓔ 90

2. In the figure below, what is the value of $\frac{b+a}{b-a}$?

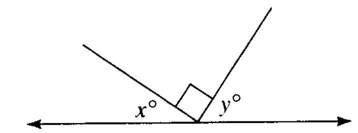


Ⓐ 1
Ⓑ 10
Ⓒ 11
Ⓓ 30
Ⓔ 36

3. In the figure below, what is the value of b?



4. In the figure below, what is the value of x if $y{:}x = 3{:}2$?



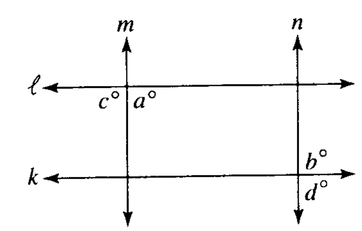
Ⓐ 18
Ⓑ 27
Ⓒ 36
Ⓓ 45
Ⓔ 54

5. What is the measure, in degrees, of the angle formed by the minute and hour hands of a clock at 1:50?

[ ] degrees

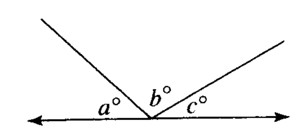
6. Concerning the figure below, if $a = b$, which of the following statements must be true?

Indicate *all* such statements.



[A] $c = d$
[B] ℓ and k are parallel
[C] m and ℓ are perpendicular

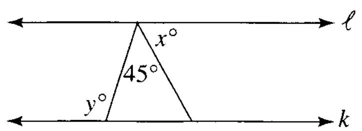
7. In the figure below, $a{:}b = 3{:}5$ and $c{:}b = 2{:}1$. What is the measure of the largest angle?



Ⓐ 30
Ⓑ 45
Ⓒ 50
Ⓓ 90
Ⓔ 100

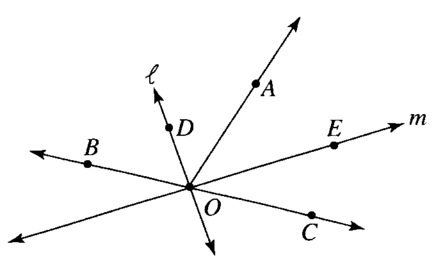
8. A, B, and C are points on a line with B between A and C. Let M and N be the midpoints of AB and BC, respectively. If $AB:BC = 3:1$, what is $MN:BC$?
   - Ⓐ 1:2
   - Ⓑ 2:3
   - Ⓒ 1:1
   - Ⓓ 3:2
   - Ⓔ 2:1

9. In the figure below, lines k and ℓ are parallel. What is the value of $y - x$?



   - Ⓐ 15
   - Ⓑ 30
   - Ⓒ 45
   - Ⓓ 60
   - Ⓔ 75

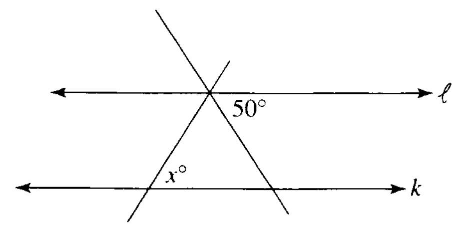
10. In the figure below, line m bisects $\angle AOC$ and line ℓ bisects $\angle AOB$. What is the measure of $\angle DOE$?



   - Ⓐ 75
   - Ⓑ 90
   - Ⓒ 100
   - Ⓓ 105
   - Ⓔ 120

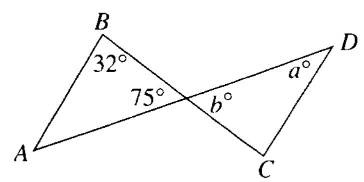
Quantitative Comparison Questions

- Ⓐ Quantity A is greater.
- Ⓑ Quantity B is greater.
- Ⓒ Quantities A and B are equal.
- Ⓓ It is impossible to determine which quantity is greater.



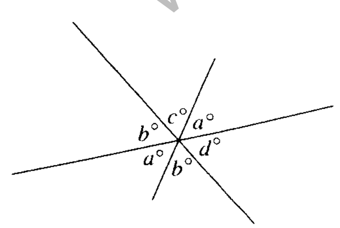
ℓ is parallel to k.

	Quantity A	Quantity B
11.	x	50

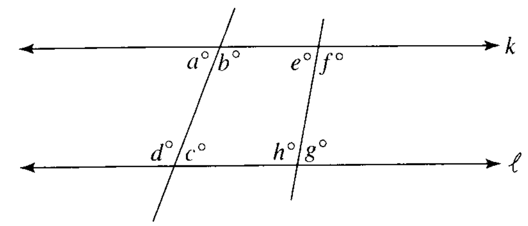


AB is parallel to CD.

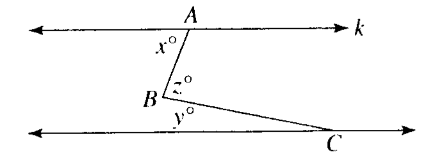
	Quantity A	Quantity B
12.	a	b



	Quantity A	Quantity B
13.	$a + b + c + d$	$2a + 2b$



	Quantity A	Quantity B
14.	$a + b + c + d$	$e + f + g + h$



k and ℓ are parallel.

	Quantity A	Quantity B
15.	z	$x + y$

ANSWER KEY

1. **D**	4. **C**	7. **E**	10. **B**	13. **D**
2. **C**	5. **115**	8. **E**	11. **D**	14. **C**
3. **36**	6. **A**	9. **C**	12. **B**	15. **C**

Answer Explanations

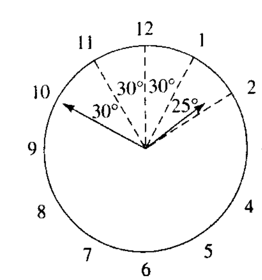
1. **(D)** The markings in the five angles are irrelevant. The sum of the measures of the five angles is 360°, and $360 \div 5 = 72$. If you calculated the measure of each angle you should have gotten 36, 54, 72, 90, and 108; but you would have wasted time.

2. **(C)** From the diagram, we see that $6a = 180$, which implies that $a = 30$, and that $5b = 180$, which implies that $b = 36$. So, $\frac{b+a}{b-a} = \frac{36+30}{36-30} = \frac{66}{6} = 11$.

3. **36** Since vertical angles are equal, the two unmarked angles are $2b$ and $4a$. Since the sum of all six angles is 360°,
$360 = 4a + 2b + 2a + 4a + 2b + b = 10a + 5b$.
However, since vertical angles are equal, $b = 2a \Rightarrow 5b = 10a$. Hence,
$360 = 10a + 5b = 10a + 10a = 20a$, so $a = 18$ and $b = 36$.

4. **(C)** Since $x + y + 90 = 180$, $x + y = 90$. Also, since $y:x = 3:2$, $y = 3t$ and $x = 2t$. Therefore, $3t + 2t = 90 \Rightarrow 5t = 90$. So $t = 18$, and $x = 2(18) = 36$.

5. **115** For problems such as this, always draw a diagram. The measure of each of the 12 central angles from one number to the next on the clock is 30°. At 1:50 the minute hand is pointing at 10, and the hour hand has gone $\frac{50}{60} = \frac{5}{6}$ of the way from 1 to 2. So from 10 to 1 on the clock is 90°, and from 1 to the hour hand is $\frac{5}{6}(30°) = 25°$, for a total of $90° + 25° = 115°$.



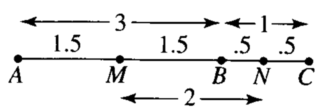
6. **(A)** No conclusions can be made about the lines; they could form any angles whatsoever. (B and C are both false.) Since $a = b$,

$$c = 180 - a = 180 - b = d.$$

(A is true.)

7. **(E)** Since $a{:}b = 3{:}5$, then $a = 3x$ and $b = 5x$. $c{:}b = c{:}5x = 2{:}1 \Rightarrow c = 10x$. Then, $3x + 5x + 10x = 180 \Rightarrow 18x = 180$. So, $x = 10$ and $c = 10x = 100$.

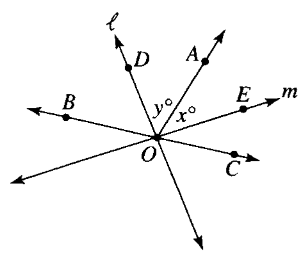
8. **(E)** If a diagram is not provided on a geometry question, draw one on your scrap paper. From the figure below, you can see that $MN{:}BC = 2{:}1$.



9. **(C)** Since the lines are parallel, the angle marked y and the sum of the angles marked x and 45 are equal: $y = x + 45 \Rightarrow y - x = 45$.

10. **(B)** Let $x = \frac{1}{2}m\angle AOC$, and $y = \frac{1}{2}m\angle AOB$.

Then, $x + y = \frac{1}{2}m\angle AOC + \frac{1}{2}m\angle AOB = \frac{1}{2}(180) = 90$.



11. **(D)** No conclusion can be made: x could equal 50 or be more or less.

12. **(B)** Since $m\angle A + 32 + 75 = 180$, $m\angle A = 73$; and since AB is parallel to CD, $a = 73$, whereas, because vertical angles are equal, $b = 75$.

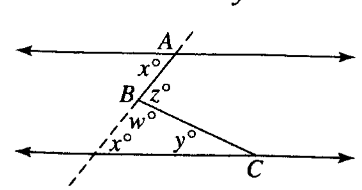
13. **(D)**

	Quantity A	Quantity B
	$a + b + c + d$	$2a + 2b$
Subtract a and b from each quantity:	$c + d$	$a + b$
Since $b = d$, subtract them:	c	a

There is no way to determine whether a is less than, greater than, or equal to c.

14. **(C)** Whether the lines are parallel or not, $a + b = c + d = e + f = g + h = 180$. Each quantity is equal to 360.

15. **(C)** Extend line segment AB to form a transversal. Since $w + z = 180$ and $w + (x + y) = 180$, it follows that $z = x + y$.



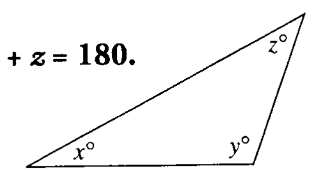
11-J. TRIANGLES

More geometry questions on the GRE pertain to triangles than to any other topic. To answer them, there are several important facts that you need to know about the angles and sides of triangles. The KEY FACTS in this section are extremely useful. Read them carefully, a few times if necessary, and *make sure you learn them all.*

KEY FACT J1

In any triangle, the sum of the measures of the three angles is 180°:

$x + y + z = 180$.



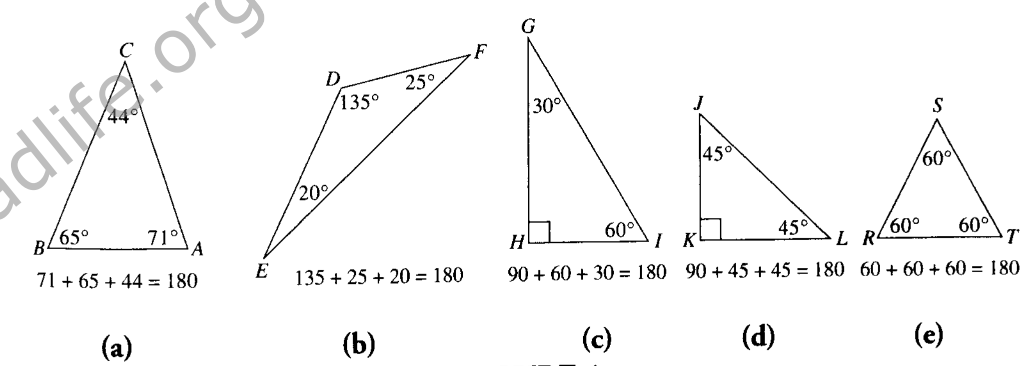
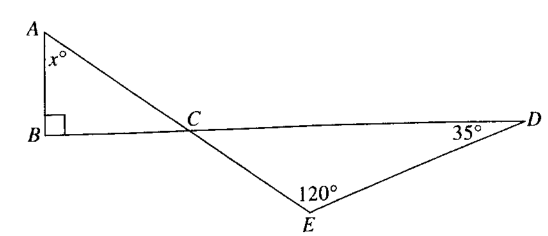


FIGURE 1

Figure 1 (a–e) illustrates KEY FACT J1 for five different triangles, which will be discussed below.

EXAMPLE 1

In the figure below, what is the value of x?



Ⓐ 25 Ⓑ 35 Ⓒ 45 Ⓓ 55 Ⓔ 65

SOLUTION.

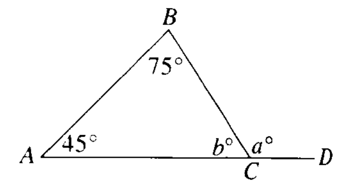
Use KEY FACT J1 twice: first, for $\triangle CDE$ and then for $\triangle ABC$.

- $m\angle DCE + 120 + 35 = 180 \Rightarrow m\angle DCE + 155 = 180 \Rightarrow m\angle DCE = 25$.
- Since vertical angles are equal, $m\angle ACB = 25$ (see KEY FACT I6).
- $x + 90 + 25 = 180 \Rightarrow x + 115 = 180$, and so $x =$ **65 (E)**.

EXAMPLE 2

In the figure at the right, what is the value of *a*?

Ⓐ 45 Ⓑ 60 Ⓒ 75 Ⓓ 120 Ⓔ 135



SOLUTION.
First find the value of b: $180 = 45 + 75 + b = 120 + b \Rightarrow b = 60$.

Then, $a + b = 180 \Rightarrow a = 180 - b = 180 - 60 =$ **120 (D)**.

In Example 2, $\angle BCD$, which is formed by one side of $\triangle ABC$ and the extension of another side, is called an ***exterior angle***. Note that to find a we did not have to first find b; we could have just added the other two angles: $a = 75 + 45 = 120$. This is a useful fact to remember.

KEY FACT J2

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles.

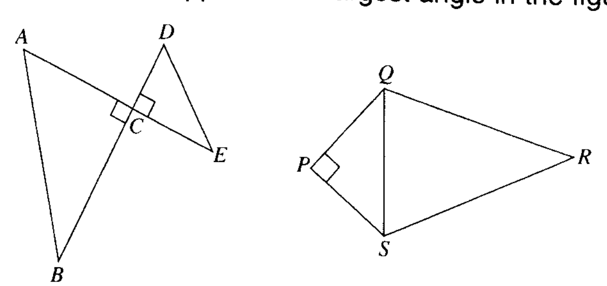
KEY FACT J3

In any triangle:

- **the longest side is opposite the largest angle;**
- **the shortest side is opposite the smallest angle;**
- **sides with the same length are opposite angles with the same measure.**

CAUTION

In KEY FACT J3 the condition "in any triangle" is crucial. If the angles are not in the same triangle, none of the conclusions hold. For example, in the figures below, *AB* and *DE* are *not* equal even though they are each opposite a 90° angle, and *QS* is not the longest side in the figure, even though it is opposite the largest angle in the figure.



Consider triangles *ABC*, *JKL*, and *RST* in Figure 1 on the previous page.

- In $\triangle ABC$: *BC* is the longest side since it is opposite angle *A*, the largest angle (71°). Similarly, *AB* is the shortest side since it is opposite angle *C*, the smallest angle (44°). So $AB < AC < BC$.

- In $\triangle JKL$: angles *J* and *L* have the same measure (45°), so $JK = KL$.
- In $\triangle RST$: since all three angles have the same measure (60°), all three sides have the same length: $RS = ST = TR$.

EXAMPLE 3

Which of the following statements concerning the length of side *YZ* is true?

Indicate *all* such statements.

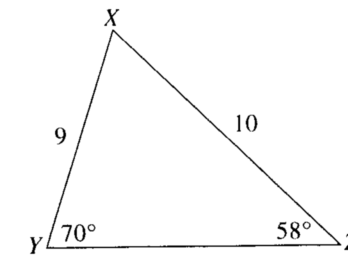
[A] $YZ < 9$

[B] $YZ = 9$

[C] $9 < YZ < 10$

[D] $YZ = 10$

[E] $YZ > 10$



SOLUTION.
Since the five answer choices are mutually exclusive, only one of them can be true.

- By KEY FACT J1, $m\angle X + 70 + 58 = 180 \Rightarrow m\angle X = 52$.
- So, *X* is the smallest angle.
- Therefore, by KEY FACT J3, *YZ* is the shortest side. So ***YZ* < 9 (A)**.

Classification of Triangles

Name	Lengths of the Sides	Measures of the Angles	Examples from Figure 1
scalene	all 3 different	all 3 different	*ABC, DEF, GHI*
isosceles	2 the same	2 the same	*JKL*
equilateral	all 3 the same	all 3 the same	*RST*

Acute triangles are triangles such as *ABC* and *RST*, in which all three angles are acute. An acute triangle could be scalene, isosceles, or equilateral.

Obtuse triangles are triangles such as *DEF*, in which one angle is obtuse and two are acute. An obtuse triangle could be scalene or isosceles.

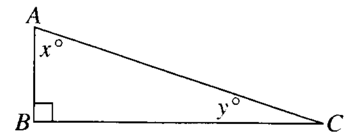
Right triangles are triangles such as *GHI* and *JKL*, which have one right angle and two acute ones. A right triangle could be scalene or isosceles. The side opposite the 90° angle is called the ***hypotenuse***, and by KEY FACT J3, it is the longest side. The other two sides are called the ***legs***.

If x and y are the measures of the acute angles of a right triangle, then by KEY FACT J1, $90 + x + y = 180 \Rightarrow x + y = 90$.

KEY FACT J4

In any right triangle, the sum of the measures of the two acute angles is 90°.

EXAMPLE 4



Quantity A	Quantity B
The average of x and y	45

TIP

The Pythagorean theorem is probably the most important theorem you need to know. Be sure to review all of its uses.

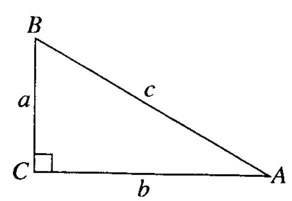
SOLUTION.
Since the diagram indicates that $\triangle ABC$ is a right triangle, then, by KEY FACT J1, $x + y = 90$. So the average of x and $y = \frac{x+y}{2} = \frac{90}{2} = 45$.

The quantities are equal (**C**).

The most important facts concerning right triangles are the **Pythagorean theorem** and its converse, which are given in KEY FACT J5 and repeated as the first line of KEY FACT J6.

KEY FACT J5

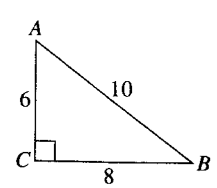
Let a, b, and c be the sides of $\triangle ABC$, with $a \leq b \leq c$. If $\triangle ABC$ is a right triangle, $a^2 + b^2 = c^2$; and if $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle.



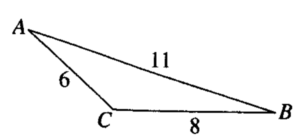
KEY FACT J6

Let a, b, and c be the sides of $\triangle ABC$, with $a \leq b \leq c$.

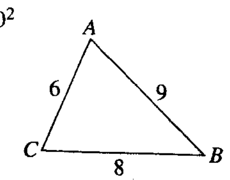
- **$a^2 + b^2 = c^2$ if and only if angle C is a right angle. ($\triangle ABC$ is a right triangle.)**
- **$a^2 + b^2 < c^2$ if and only if angle C is obtuse. ($\triangle ABC$ is an obtuse triangle.)**
- **$a^2 + b^2 > c^2$ if and only if angle C is acute. ($\triangle ABC$ is an acute triangle.)**



$6^2 + 8^2 = 10^2$



$6^2 + 8^2 < 11^2$



$6^2 + 8^2 > 9^2$

EXAMPLE 5

Which of the following triples are *not* the sides of a right triangle? Indicate *all* such triples.

- [A] 3, 4, 5
- [B] 1, 1, $\sqrt{3}$
- [C] 1, $\sqrt{3}$, 2
- [D] $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$
- [E] 30, 40, 50

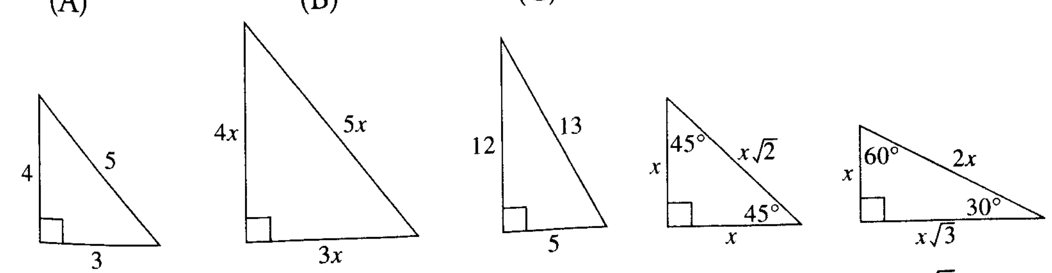
SOLUTION.
Just check each choice.

(A) $3^2 + 4^2 = 9 + 16 = 25 = 5^2$	These *are* the sides of a right triangle.
(B) $1^2 + 1^2 = 1 + 1 = 2 \neq (\sqrt{3})^2$	These *are not* the sides of a right triangle.
(C) $1^2 + (\sqrt{3})^2 = 1 + 3 = 4 = 2^2$	These *are* the sides of a right triangle.
(D) $(\sqrt{3})^2 + (\sqrt{4})^2 = 3 + 4 = 7 \neq (\sqrt{5})^2$	These *are not* the sides of a right triangle.
(E) $30^2 + 40^2 = 900 + 1600 = 2500 = 50^2$	These *are* the sides of a right triangle.

The answer is **B** and **D**.

Below are the right triangles that appear most often on the GRE. You should recognize them immediately whenever they come up in questions. Carefully study each one, and memorize KEY FACTS J7–J11.

(A) (B) (C) (D) (E)



3, 4, 5 — $3x$, $4x$, $5x$ — 5, 12, 13 — x, x, $x\sqrt{2}$ — x, $x\sqrt{3}$, $2x$

On the GRE, the most common right triangles whose sides are integers are the 3-4-5 triangle (A) and its multiples (B).

KEY FACT J7

For any positive number x, there is a right triangle whose sides are $3x$, $4x$, $5x$.

For example:

$x = 1$	3, 4, 5	$x = 5$	15, 20, 25
$x = 2$	6, 8, 10	$x = 10$	30, 40, 50
$x = 3$	9, 12, 15	$x = 50$	150, 200, 250
$x = 4$	12, 16, 20	$x = 100$	300, 400, 500

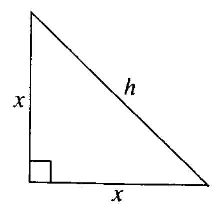
TIP

KEY FACT J7 applies even if x is not an integer. For example:
$x = .5$ 1.5, 2, 2.5
$x = \pi$ 3π, 4π, 5π

NOTE: The only other right triangle with integer sides that you should recognize immediately is the one whose sides are 5, 12, 13 (C).

Let x = length of each leg, and h = length of the hypotenuse, of an isosceles right triangle (D). By the Pythagorean theorem (KEY FACT J5), $x^2 + x^2 = h^2$.

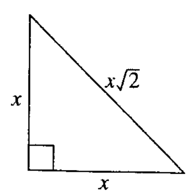
So, $2x^2 = h^2$, and $h = \sqrt{2x^2} = x\sqrt{2}$.

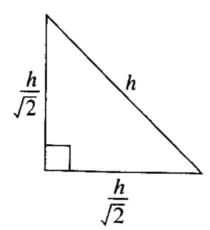


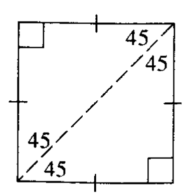
KEY FACT J8

In a 45-45-90 right triangle, the sides are x, x, and $x\sqrt{2}$. So,

- **by multiplying the length of a leg by $\sqrt{2}$, you get the hypotenuse.**
- **by dividing the hypotenuse by $\sqrt{2}$, you get the length of each leg.**







KEY FACT J9

The diagonal of a square divides the square into two isosceles right triangles.

The last important right triangle is the one whose angles measure 30°, 60°, and 90°. (E)

KEY FACT J10

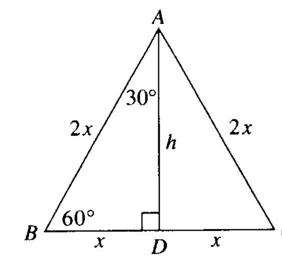
An altitude divides an equilateral triangle into two 30-60-90 right triangles.

Let $2x$ be the length of each side of equilateral $\triangle ABC$ in which altitude AD is drawn. Then $\triangle ABD$ is a 30-60-90 right triangle, and its sides are x, $2x$, and h.

By the Pythagorean theorem,

$x^2 + h^2 = (2x)^2 = 4x^2$.

So $h^2 = 3x^2$, and $h = \sqrt{3x^2} = x\sqrt{3}$.



KEY FACT J11

In a 30-60-90 right triangle the sides are x, $x\sqrt{3}$, and $2x$.

If you know the length of the shorter leg (x),

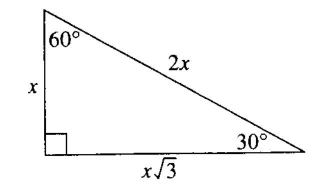
- multiply it by $\sqrt{3}$ to get the longer leg, and
- multiply it by 2 to get the hypotenuse.

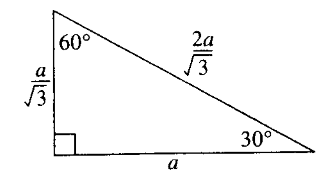
If you know the length of the longer leg (a),

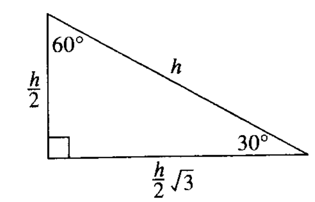
- divide it by $\sqrt{3}$ to get the shorter leg, and
- multiply the shorter leg by 2 to get the hypotenuse.

If you know the length of the hypotenuse (h),

- divide it by 2 to get the shorter leg, and
- multiply the shorter leg by $\sqrt{3}$ to get the longer leg.







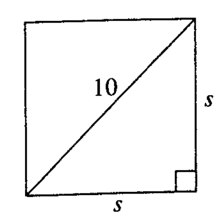
EXAMPLE 6

What is the area of a square whose diagonal is 10?

SOLUTION.

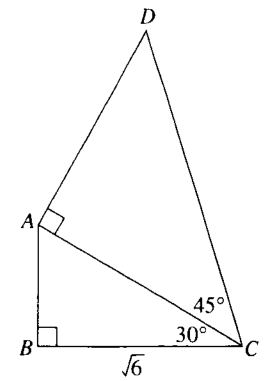
Draw a diagonal in a square of side s, creating a 45-45-90 right triangle. By KEY FACT J8:

$s = \frac{10}{\sqrt{2}}$ and $A = s^2 = \left(\frac{10}{\sqrt{2}}\right)^2 = \frac{100}{2} = \mathbf{50}$.



EXAMPLE 7

In the diagram at the right, if $BC = \sqrt{6}$, what is the value of CD?



Ⓐ $2\sqrt{2}$

Ⓑ $4\sqrt{2}$

Ⓒ $2\sqrt{3}$

Ⓓ $2\sqrt{6}$

Ⓔ 4

SOLUTION.
Since $\triangle ABC$ and $\triangle DAC$ are 30-60-90 and 45-45-90 right triangles, respectively, use KEY FACTS J11 and J8.

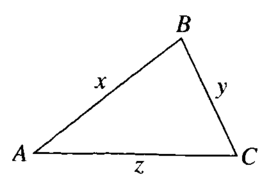
- Divide the longer leg, BC, by $\sqrt{3}$ to get the shorter leg, AB: $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$.
- Multiply AB by 2 to get the hypotenuse: $AC = 2\sqrt{2}$.
- Since AC is also a leg of isosceles right $\triangle DAC$, to get hypotenuse CD, multiply AC by $\sqrt{2}$: $CD = 2\sqrt{2} \times \sqrt{2} = 2 \times 2 = \mathbf{4}$ **(E)**.

Key Fact J12

Triangle Inequality

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

The best way to remember this is to see that $x + y$, the length of the path from A to C through B, is greater than z, the length of the direct path from A to C.



$x + y > z$

NOTE: If you subtract x from each side of $x + y > z$, you see that $z - x < y$.

KEY FACT J13

The difference of the lengths of any two sides of a triangle is less than the length of the third side.

EXAMPLE 8

If the lengths of two of the sides of a triangle are 6 and 7, which of the following could be the length of the third side?

Indicate *all* possible lengths.

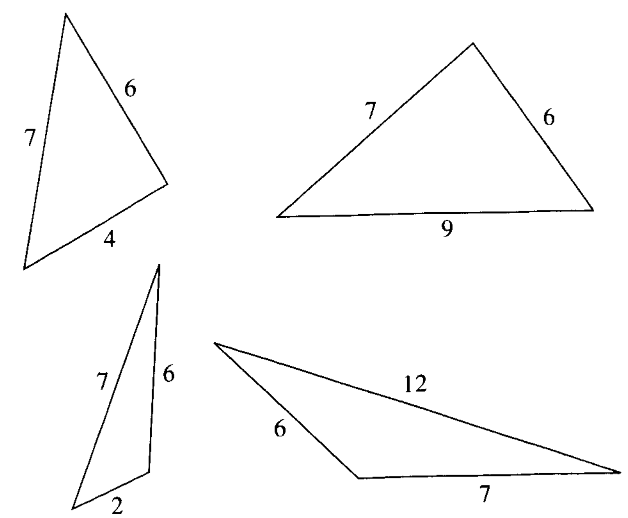
[A] 1	[C] π	[E] 12	[G] 15
[B] 2	[D] 7	[F] 13	

SOLUTION.
Use KEY FACTS J12 and J13.

- The third side must be *less* than 6 + 7 = 13. (Eliminate F and G.)
- The third side must be *greater* than 7 − 6 = 1. (Eliminate A.)
- *Any* number between 1 and 13 could be the length of the third side.

The answer is **B, C, D, E.**

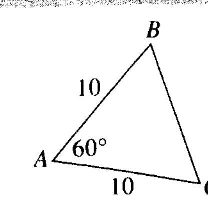
The following diagram illustrates several triangles, two of whose sides have lengths of 6 and 7.



On the GRE, two other terms that appear regularly in triangle problems are ***perimeter*** and ***area*** (see Section 11-K).

EXAMPLE 9

In the figure at the right, what is the perimeter of $\triangle ABC$?



Ⓐ $20 + 10\sqrt{2}$

Ⓑ $20 + 10\sqrt{3}$

Ⓒ 25

Ⓓ 30

Ⓔ 40

SOLUTION.
First, use KEY FACTS J3 and J1 to find the measures of the angles.

- Since $AB = AC$, $m\angle B = m\angle C$. Represent each of them by x.
- Then, $x + x + 60 = 180 \Rightarrow 2x = 120 \Rightarrow x = 60$.
- Since the measure of each angle of $\triangle ABC$ is 60, the triangle is equilateral.
- So $BC = 10$, and the perimeter is $10 + 10 + 10 =$ **30 (D)**.

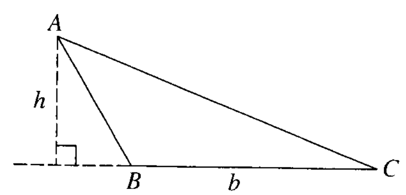
KEY FACT J14

The area of a triangle is given by $A = \frac{1}{2}bh$, where b is the base and h is the height.

NOTE:

1. *Any* side of the triangle can be taken as the ***base.***
2. The ***height*** or ***altitude*** is a line segment drawn to the base or, if necessary, to an extension of the base from the opposite vertex.
3. In a right triangle, either leg can be the base and the other the height.
4. The height may be outside the triangle. [See the figure below.]

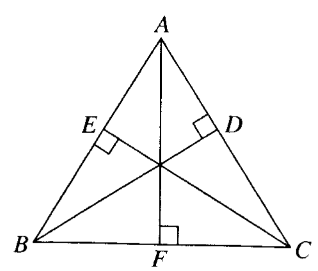
TIP
If one endpoint of the base of a triangle is the vertex of an obtuse angle, then the height drawn to that base will be outside the triangle.



Note: $\triangle ABC$ is obtuse.

In the figure below:

- If AC is the base, BD is the height.
- If AB is the base, CE is the height.
- If BC is the base, AF is the height.

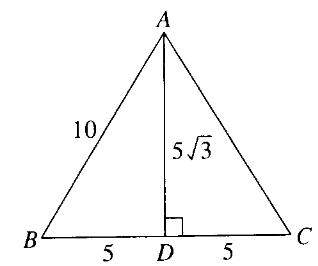


EXAMPLE 10

What is the area of an equilateral triangle whose sides are 10?

Ⓐ 30 Ⓑ $25\sqrt{3}$ Ⓒ 50 Ⓓ $50\sqrt{3}$ Ⓔ 100

SOLUTION.
Draw an equilateral triangle and one of its altitudes.



- By KEY FACT J10, $\triangle ABD$ is a 30-60-90 right triangle.
- By KEY FACT J11, $BD = 5$ and $AD = 5\sqrt{3}$.
- The area of $\triangle ABC = \frac{1}{2}(10)(5\sqrt{3}) =$ **$25\sqrt{3}$ (B)**.

Replacing 10 by s in Example 10 yields a very useful result.

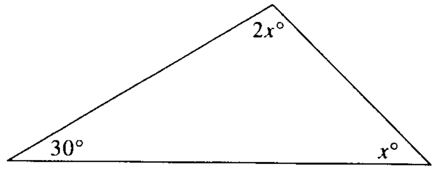
KEY FACT J15

If A represents the area of an equilateral triangle with side s, then $A = \frac{s^2\sqrt{3}}{4}$.

TIP
Learn this formula for the area of an equilateral triangle. It can save you time.

Practice Exercises — Triangles

Discrete Quantitative Questions



1. In the triangle above, what is the value of x?
   (A) 20
   (B) 30
   (C) 40
   (D) 50
   (E) 60

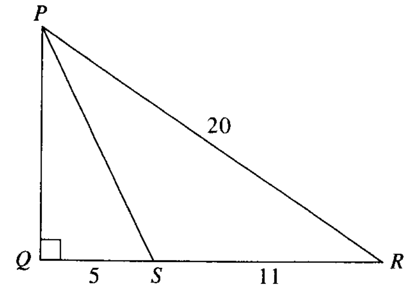
2. If the difference between the measures of the two smaller angles of a right triangle is 8°, what is the measure, in degrees, of the smallest angle?
   (A) 37
   (B) 41
   (C) 42
   (D) 49
   (E) 53

3. What is the area of an equilateral triangle whose altitude is 6?
   (A) 18
   (B) $12\sqrt{3}$
   (C) $18\sqrt{3}$
   (D) 36
   (E) $24\sqrt{3}$

4. Two sides of a right triangle are 12 and 13. Which of the following *could be* the length of the third side?

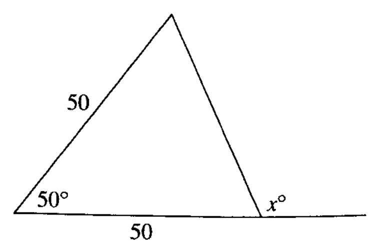
   Indicate *all* possible lengths.

   [A] 2
   [B] 5
   [C] $\sqrt{31}$
   [D] 11
   [E] $\sqrt{313}$

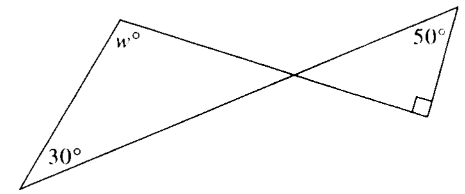


5. What is the value of PS in the triangle above?
   (A) $5\sqrt{2}$
   (B) 10
   (C) 11
   (D) 13
   (E) $12\sqrt{2}$

6. If the measures of the angles of a triangle are in the ratio of 1:2:3, and if the length of the smallest side of the triangle is 10, what is the length of the longest side?
   (A) $10\sqrt{2}$
   (B) $10\sqrt{3}$
   (C) 15
   (D) 20
   (E) 30

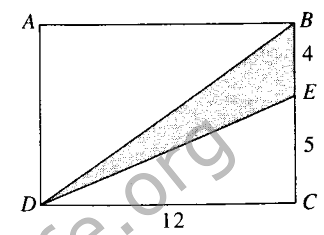


7. What is the value of x in the figure above?



8. In the figure above, what is the value of w?

Questions 9–10 refer to the following figure.

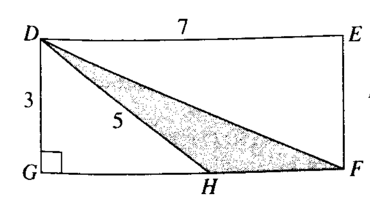


ABCD is a rectangle.

9. What is the area of $\triangle BED$?
   (A) 12
   (B) 24
   (C) 36
   (D) 48
   (E) 60

10. What is the perimeter of $\triangle BED$?
    (A) $19 + 5\sqrt{2}$
    (B) 28
    (C) $17 + \sqrt{185}$
    (D) 32
    (E) 36

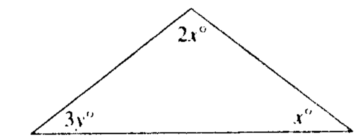
Questions 11–12 refer to the following figure.



DEFG is a rectangle.

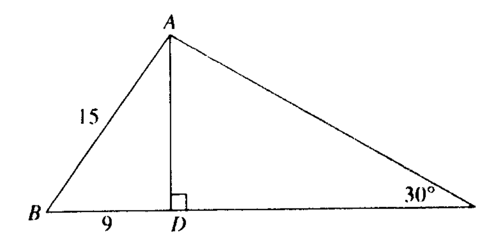
11. What is the area of $\triangle DFH$?
    (A) 3
    (B) 4.5
    (C) 6
    (D) 7.5
    (E) 10

12. What is the perimeter of $\triangle DFH$?
    (A) $8 + \sqrt{41}$
    (B) $8 + \sqrt{58}$
    (C) 16
    (D) 17
    (E) 18



13. Which of the following expresses a true relationship between x and y in the figure above?
    (A) $y = 60 - x$
    (B) $y = x$
    (C) $x + y = 90$
    (D) $y = 180 - 3x$
    (E) $x = 90 - 3y$

Questions 14–15 refer to the following figure.



14. What is the perimeter of $\triangle ABC$?
    (A) 48
    (B) $48 + 12\sqrt{2}$
    (C) $48 + 12\sqrt{3}$
    (D) 60
    (E) $60 + 6\sqrt{3}$

15. What is the area of $\triangle ABC$?
    (A) 108
    (B) $54 + 72\sqrt{2}$
    (C) $54 + 72\sqrt{3}$
    (D) 198
    (E) 216

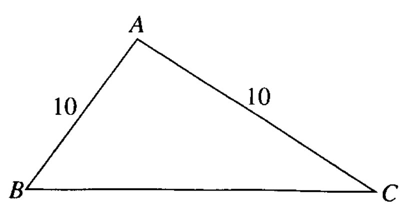
Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

The lengths of two sides of a triangle are 7 and 11.

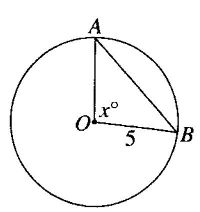
	Quantity A	Quantity B
16.	The length of the third side	4

	Quantity A	Quantity B
17.	The ratio of the length of a diagonal to the length of a side of a square	$\sqrt{2}$



	Quantity A	Quantity B
18.	The perimeter of $\triangle ABC$	30

Questions 19–20 refer to the following figure.



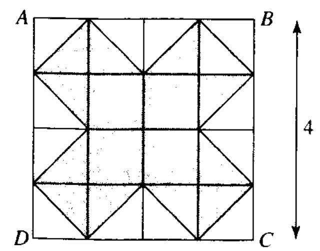
$90 < x$

	Quantity A	Quantity B
19.	The length of AB	7

	Quantity A	Quantity B
20.	The perimeter of $\triangle AOB$	20

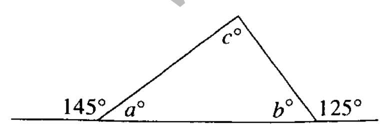
	Quantity A	Quantity B
21.	The area of an equilateral triangle whose sides are 10	The area of an equilateral triangle whose altitude is 10

Questions 22–23 refer to the following figure in which the horizontal and vertical lines divide square *ABCD* into 16 smaller squares.

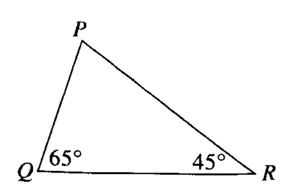


	Quantity A	Quantity B
22.	The perimeter of the shaded region	The perimeter of the square

	Quantity A	Quantity B
23.	The area of the shaded region	The area of the white region



	Quantity A	Quantity B
24.	$a + b$	c



	Quantity A	Quantity B
25.	PR	QR

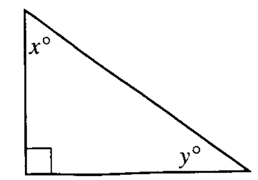
ANSWER KEY

1. **D**	6. **D**	11. **B**	16. **A**	21. **B**
2. **B**	7. **115**	12. **B**	17. **C**	22. **A**
3. **B**	8. **110**	13. **A**	18. **D**	23. **A**
4. **B, E**	9. **B**	14. **C**	19. **A**	24. **C**
5. **D**	10. **D**	15. **C**	20. **B**	25. **B**

Answer Explanations

1. **(D)** $x + 2x + 30 = 180 \Rightarrow 3x + 30 = 180 \Rightarrow 3x = 150 \Rightarrow x = 50$.

2. **(B)** Draw a diagram and label it.



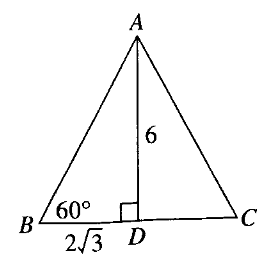
Then write the equations: $x + y = 90$ and $x - y = 8$.

Add the equations:

$$\begin{array}{r} x + y = 90 \\ + \; x - y = \;\; 8 \\ \hline 2x = 98 \end{array}$$

So $x = 49$ and $y = 90 - 49 = 41$.

3. **(B)** Draw altitude *AD* in equilateral $\triangle ABC$.



By KEY FACT J11, $BD = \dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$, and *BD* is one half the base.

So, the area is $2\sqrt{3} \times 6 = 12\sqrt{3}$.

4. **(B)(E)** If the triangle were not required to be a right triangle, by KEY FACTS J11 and J12 *any* number greater than 1 and less than 25 could be the length of the third side, and the answer would be A, B, C, D, E. But for a right triangle, there are only *two* possibilities:
   - If 13 is the hypotenuse, then the legs are 12 and 5. (B is true.) (If you didn't recognize the 5-12-13 triangle, use the Pythagorean theorem: $12^2 + x^2 = 13^2$, and solve.)
   - If 12 and 13 are the two legs, then use the Pythagorean theorem to find the hypotenuse: $12^2 + 13^2 = c^2 \Rightarrow c^2 = 144 + 169 = 313 \Rightarrow c = \sqrt{313}$. (E is true.)

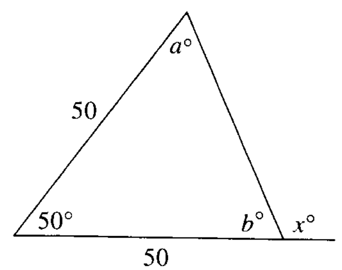
5. **(D)** Use the Pythagorean theorem twice, unless you recognize the common right triangles in this figure (*which you should*). Since $PR = 20$ and $QR = 16$, $\triangle PQR$ is a $3x$-$4x$-$5x$ right triangle with $x = 4$. So $PQ = 12$, and $\triangle PQS$ is a right triangle whose legs are 5 and 12. The hypotenuse, PS, therefore, is 13.

6. **(D)** If the measures of the angles are in the ratio of 1:2:3,

   $x + 2x + 3x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30$.

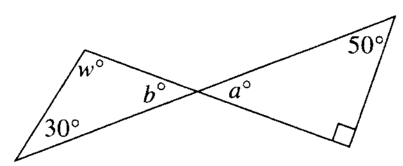
   So the triangle is a 30-60-90 right triangle, and the sides are a, $2a$, and $a\sqrt{3}$. Since $a = 10$, then $2a$, the length of the longest side, is 20.

7. **115** Label the other angles in the triangle.



   $50 + a + b = 180 \Rightarrow a + b = 130$, and since the triangle is isosceles, $a = b$. Therefore, a and b are each 65, and $x = 180 - 65 = 115$.

8. **110** Here, $50 + 90 + a = 180 \Rightarrow a = 40$, and since vertical angles are equal, $b = 40$. Then, $40 + 30 + w = 180 \Rightarrow w = 110$.



9. **(B)** You *could* calculate the area of the rectangle and subtract the area of the two white right triangles, but you shouldn't. It is easier to solve this problem if you realize that the shaded area is a triangle whose base is 4 and whose height is 12. The area is $\frac{1}{2}(4)(12) = 24$.

10. **(D)** Since both BD and ED are the hypotenuses of right triangles, their lengths can be calculated by the Pythagorean theorem, but these are triangles you should recognize: the sides of $\triangle DCE$ are 5-12-13, and those of $\triangle BAD$ are 9-12-15 ($3x$-$4x$-$5x$, with $x = 3$). So the perimeter of $\triangle BED$ is $4 + 13 + 15 = 32$.

11. **(B)** Since $\triangle DGH$ is a right triangle whose hypotenuse is 5 and one of whose legs is 3, the other leg, GH, is 4. Since $GF = DE$ is 7, HF is 3. Now, $\triangle DFH$ has a base of 3 (HF) and a height of 3 (DG), and its area is $\frac{1}{2}(3)(3) = 4.5$.

12. **(B)** In $\triangle DFH$, we already have that $DH = 5$ and $HF = 3$; we need only find DF, which is the hypotenuse of $\triangle DEF$. By the Pythagorean theorem,

    $(DF)^2 = 3^2 + 7^2 = 9 + 49 = 58 \Rightarrow DF = \sqrt{58}$.

    So the perimeter is $3 + 5 + \sqrt{58} = 8 + \sqrt{58}$.

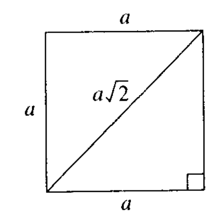
13. **(A)** $x + 2x + 3y = 180 \Rightarrow 3x + 3y = 180$. So $x + y = 60$, and $y = 60 - x$.

14. **(C)** $\triangle ABD$ is a right triangle whose hypotenuse is 15 and one of whose legs is 9, so this is a $3x$-$4x$-$5x$ triangle with $x = 3$; so $AD = 12$. Now $\triangle ADC$ is a 30-60-90 triangle, whose shorter leg is 12. Hypotenuse AC is 24, and leg CD is $12\sqrt{3}$. So the perimeter is $24 + 15 + 9 + 12\sqrt{3} = 48 + 12\sqrt{3}$.

15. **(C)** From the solution to 14, we have the base $(9 + 12\sqrt{3})$ and the height (12) of $\triangle ABC$. Then, the area is $\frac{1}{2}(12)(9 + 12\sqrt{3}) = 54 + 72\sqrt{3}$.

16. **(A)** Any side of a triangle must be greater than the difference of the other two sides (KEY FACT J13), so the third side is greater than $11 - 7 = 4$.

17. **(C)** Draw a diagram. A diagonal of a square is the hypotenuse of each of the two 45-45-90 right triangles formed. The ratio of the length of the hypotenuse to the length of the leg in such a triangle is $\sqrt{2}:1$, so the quantities are equal.

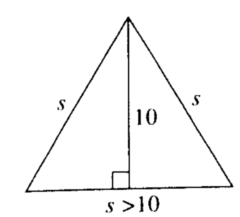


18. **(D)** BC can be any positive number less than 20 (by KEY FACTS J12 and J13, $BC > 10 - 10 = 0$ and $BC < 10 + 10 = 20$). So the perimeter can be *any* number greater than 20 and less than 40.

19. **(A)** Since OA and OB are radii, they are each equal to 5. With no restrictions on x, AB could be any positive number less than 10, and the bigger x is, the bigger AB is. If x were 90, AB would be $5\sqrt{2}$, but we are told that $x > 90$, so $AB > 5\sqrt{2} > 7$.

20. **(B)** Since AB must be less than 10, the perimeter is *less* than 20.

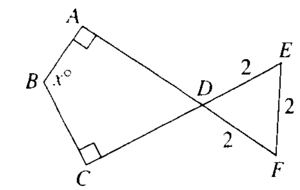
21. **(B)** Don't calculate either area. The length of a side of an equilateral triangle is *greater* than the length of an altitude. So Quantity B is larger since it is the area of a triangle whose sides are greater.



EXAMPLE 1

In the figure at the right, what is the value of x?

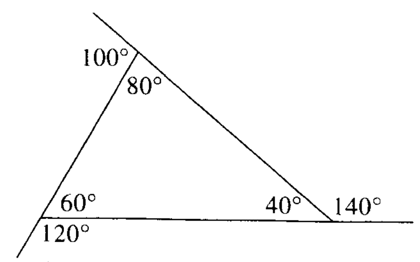
Ⓐ 60 Ⓑ 90 Ⓒ 100 Ⓓ 120 Ⓔ 150



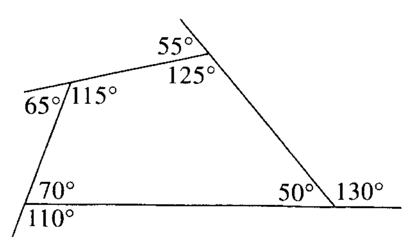
SOLUTION.

Since $\triangle DEF$ is equilateral, all of its angles measure 60°; also, since the two angles at vertex D are vertical angles, their measures are equal. Therefore, the measure of $\angle D$ in quadrilateral $ABCD$ is 60°. Finally, since the sum of the measures of all four angles of $ABCD$ is 360°, $60 + 90 + 90 + x = 360 \Rightarrow 240 + x = 360 \Rightarrow x =$ **120 (D)**.

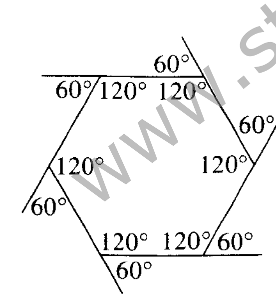
In the polygons in the figure that follows, one exterior angle has been drawn at each vertex. Surprisingly, if you add the measures of all of the exterior angles in any of the polygons, the sums are equal.



100 + 120 + 140 = 360



65 + 110 + 130 + 55 = 360



60 + 60 + 60 + 60 + 60 + 60 = 360

KEY FACT K3

In any polygon, the sum of the exterior angles, taking one at each vertex, is 360°.

A ***regular polygon*** is a polygon in which all of the sides are the same length and each angle has the same measure. KEY FACT K4 follows immediately from this definition and from KEY FACTS K2 and K3.

KEY FACT K4

In any regular polygon the measure of each interior angle is $\dfrac{(n-2)\times 180°}{n}$ **and**

EXAMPLE 2

What is the measure, in degrees, of each interior angle in a regular decagon?

[ ] degrees

SOLUTION 1.

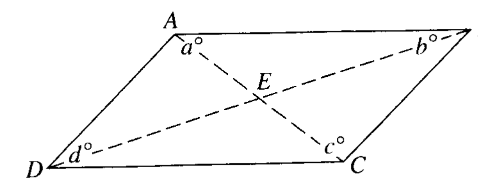
The measure of each of the 10 interior angles is

$$\frac{(10-2)\times 180°}{10} = \frac{8\times 180°}{10} = \frac{1440°}{10} = \mathbf{144°}.$$

SOLUTION 2.

The measure of each of the 10 exterior angles is 36° (360° ÷ 10). Therefore, the measure of each interior angle is 180° − 36° = **144°**.

A ***parallelogram*** is a quadrilateral in which both pairs of opposite sides are parallel.

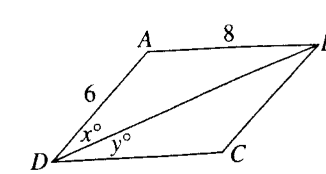


KEY FACT K5

Parallelograms have the following properties:

- **Opposite sides are equal: $AB = CD$ and $AD = BC$.**
- **Opposite angles are equal: $a = c$ and $b = d$.**
- **Consecutive angles add up to 180°: $a + b = 180$, $b + c = 180$, $c + d = 180$, and $a + d = 180$.**
- **The two diagonals bisect each other: $AE = EC$ and $BE = ED$.**
- **A diagonal divides the parallelogram into two triangles that have the exact same size and shape. (The triangles are congruent.)**

EXAMPLE 3

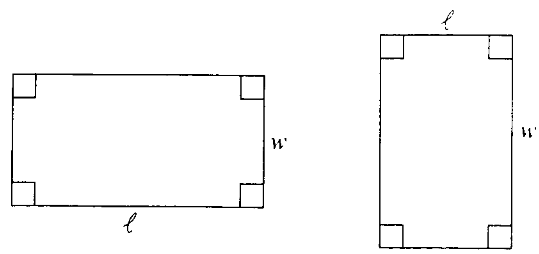


ABCD is a parallelogram.

Quantity A	Quantity B
x	y

SOLUTION.
In $\triangle ABD$ the larger angle is opposite the larger side (KEY FACT J2); so $x > m\angle ABD$. However, since AB and CD are parallel sides cut by transversal BD, $y = m\angle ABD$. Therefore, $x > y$. Quantity **A** is greater.

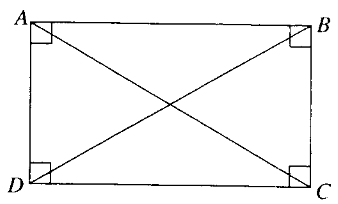
A ***rectangle*** is a parallelogram in which all four angles are right angles. Two adjacent sides of a rectangle are usually called the ***length*** (ℓ) and the ***width*** (w). Note in the right-hand figure that the length is not necessarily greater than the width.



KEY FACT K6

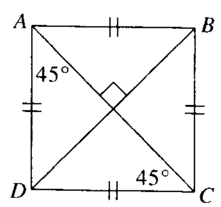
Since a rectangle is a parallelogram, all of the properties listed in KEY FACT K5 hold for rectangles. In addition:

- **The measure of each angle in a rectangle is 90°.**
- **The diagonals of a rectangle have the same length: $AC = BD$.**



TIP
A rectangle is a parallelogram.

A ***square*** is a rectangle in which all four sides have the same length.



KEY FACT K7

Since a square is a rectangle, all of the properties listed in KEY FACTS K5 and K6 hold for squares. In addition:

- **All four sides have the same length.**
- **Each diagonal divides the square into two 45-45-90 right triangles.**
- **The diagonals are perpendicular to each other: $AC \perp BD$.**

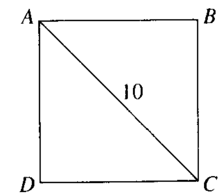
TIP
A square is a rectangle and, hence, a parallelogram.

EXAMPLE 4

What is the length of each side of a square if its diagonals are 10?

Ⓐ 5 Ⓑ 7 Ⓒ $5\sqrt{2}$ Ⓓ $10\sqrt{2}$ Ⓔ $10\sqrt{3}$

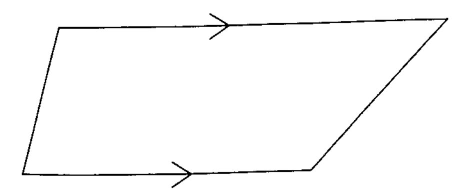
SOLUTION.
Draw a diagram.



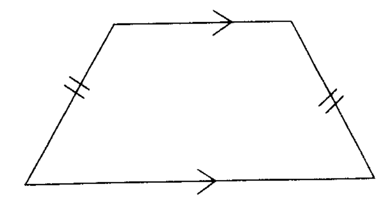
In square $ABCD$, diagonal AC is the hypotenuse of $\triangle ABC$ a 45-45-90 right triangle, and side AB is a leg of that triangle. By KEY FACT J7,

$$AB = \frac{AC}{\sqrt{2}} = \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \text{ (C)}.$$

A ***trapezoid*** is a quadrilateral in which one pair of sides is parallel and the other pair of sides is not parallel. The parallel sides are called the ***bases*** of the trapezoid. The two bases are never equal. In general, the two nonparallel sides are not equal; if they are the trapezoid is called an ***isosceles trapezoid***.

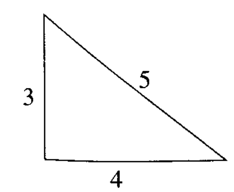


Trapezoid

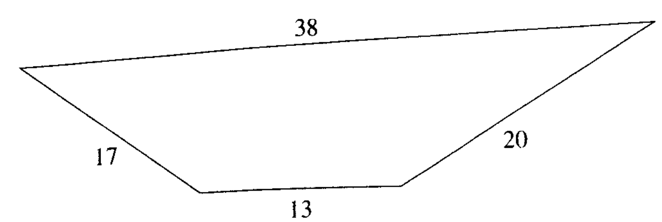


Isoceles trapezoid

The ***perimeter*** (P) of any polygon is the sum of the lengths of all of its sides.



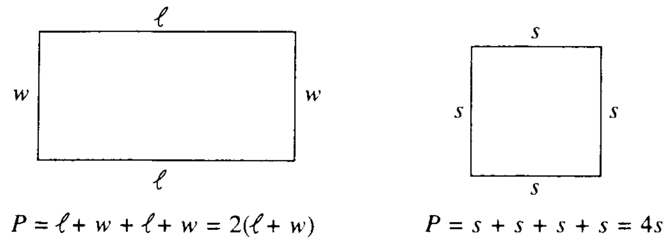
$P = 3 + 4 + 5 = 12$



$P = 13 + 17 + 20 + 38 = 88$

KEY FACT K8

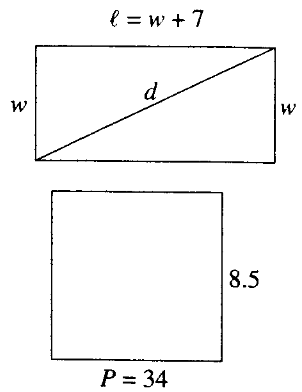
In a rectangle, $P = 2(\ell + w)$; in a square, $P = 4s$.



EXAMPLE 5

The length of a rectangle is 7 more than its width. If the perimeter of the rectangle is the same as the perimeter of a square of side 8.5, what is the length of a diagonal of the rectangle?

SOLUTION.
Don't do anything until you have drawn a diagram.



Since the perimeter of the square = $4 \times 8.5 = 34$, the perimeter of the rectangle is also 34: $2(\ell + w) = 34 \Rightarrow \ell + w = 17$. Replacing ℓ by $w + 7$, we get:

$$w + 7 + w = 17 \Rightarrow 2w + 7 = 17 \Rightarrow 2w = 10 \Rightarrow w = 5$$

Then $\ell = 5 + 7 = 12$. Finally, realize that the diagonal is the hypotenuse of a 5-12-13 triangle, or use the Pythagorean theorem:

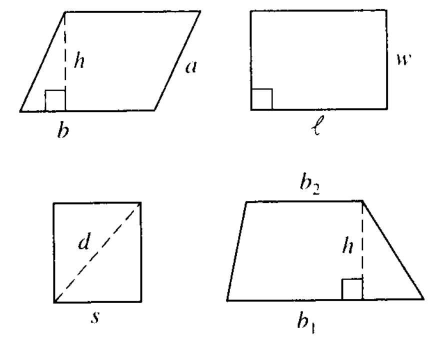
$$d^2 = 5^2 + 12^2 = 25 + 144 = 169 \Rightarrow d = \mathbf{13}.$$

In Section 11-J we reviewed the formula for the area of a triangle. The only other figures for which you need to know area formulas are the parallelogram, rectangle, square, and trapezoid.

KEY FACT K9

Here are the area formulas you need to know:

- **For a parallelogram: $A = bh$.**
- **For a rectangle: $A = \ell w$.**
- **For a square: $A = s^2$ or $A = \frac{1}{2}d^2$.**
- **For a trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$.**

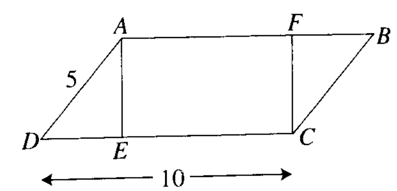


TIP

Be sure to learn the alternative formula for the area of a square: $A = 1/2d^2$, where d is the length of a diagonal.

EXAMPLE 6

In the figure below, the area of parallelogram *ABCD* is 40. What is the area of rectangle *AFCE*?

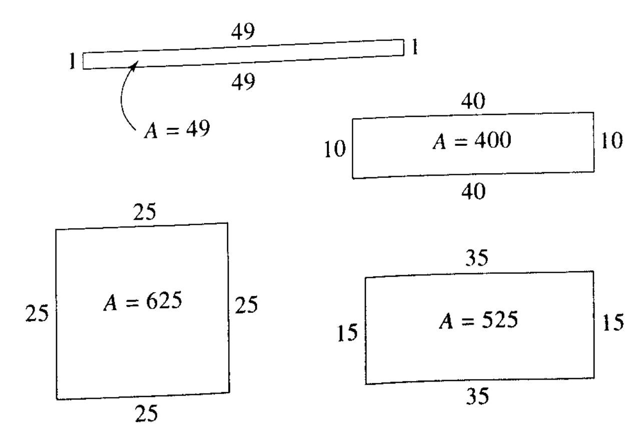


Ⓐ 20 Ⓑ 24 Ⓒ 28 Ⓓ 32 Ⓔ 36

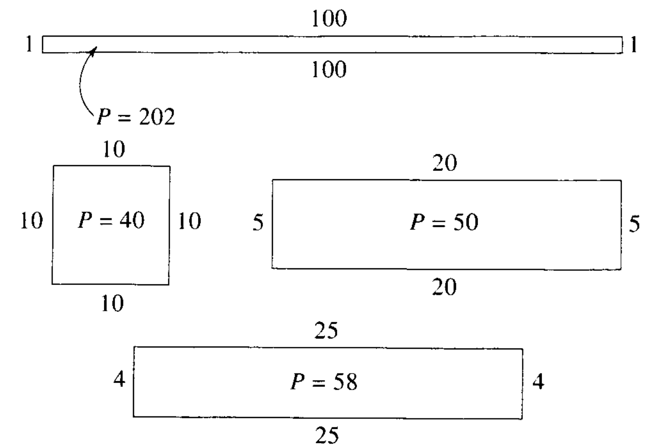
SOLUTION.
Since the base, *CD*, is 10 and the area is 40, the height, *AE*, must be 4. Then $\triangle AED$ must be a 3-4-5 right triangle with $DE = 3$, which implies that $EC = 7$. So the area of the rectangle is $7 \times 4 = \mathbf{28}$ **(C)**.

Two rectangles with the same perimeter can have different areas, and two rectangles with the same area can have different perimeters. These facts are a common source of questions on the GRE.

RECTANGLES WHOSE PERIMETERS ARE 100



RECTANGLES WHOSE AREAS ARE 100



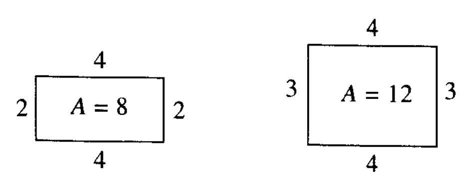
KEY FACT K10

For a given perimeter, the rectangle with the largest area is a square. For a given area, the rectangle with the smallest perimeter is a square.

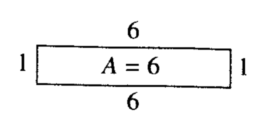
EXAMPLE 7

Quantity A	Quantity B
The area of a rectangle whose perimeter is 12	The area of a rectangle whose perimeter is 14

SOLUTION.
Draw any rectangles whose perimeters are 12 and 14 and compute their areas. As drawn below, Quantity A = 8 and Quantity B = 12.



This time Quantity B is greater. Is it always? Draw a different rectangle whose perimeter is 14.



The one drawn here has an area of 6. Now Quantity B isn't greater. The answer is **D**.

EXAMPLE 8

Quantity A	Quantity B
The area of a rectangle whose perimeter is 12	10

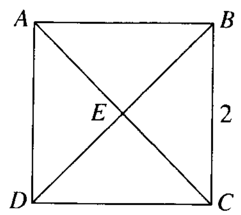
SOLUTION.
There are many rectangles of different areas whose perimeters are 12. But the largest area is 9, when the rectangle is a 3×3 square. Quantity **B** is greater.

Practice Exercises — Quadrilaterals

Discrete Quantitative Questions

1. If the length of a rectangle is 4 times its width, and if its area is 144, what is its perimeter?

Questions 2–3 refer to the diagram below in which the diagonals of square *ABCD* intersect at *E*.

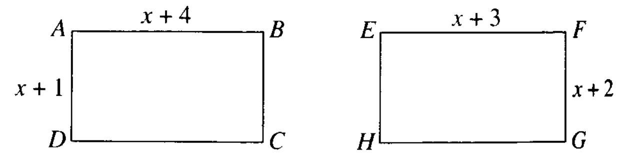


2. What is the area of $\triangle DEC$?
   - Ⓐ $\frac{1}{2}$
   - Ⓑ 1
   - Ⓒ $\sqrt{2}$
   - Ⓓ 2
   - Ⓔ $2\sqrt{2}$

3. What is the perimeter of $\triangle DEC$?
   - Ⓐ $1 + \sqrt{2}$
   - Ⓑ $2 + \sqrt{2}$
   - Ⓒ 4
   - Ⓓ $2 + 2\sqrt{2}$
   - Ⓔ 6

4. If the angles of a five-sided polygon are in the ratio of 2:3:3:5:5, what is the measure of the smallest angle?
   - Ⓐ 20
   - Ⓑ 40
   - Ⓒ 60
   - Ⓓ 80
   - Ⓔ 90

5. If in the figures below, the area of rectangle *ABCD* is 100, what is the area of rectangle *EFGH*?



   - Ⓐ 98
   - Ⓑ 100
   - Ⓒ 102
   - Ⓓ 104
   - Ⓔ 106

Questions 6–7 refer to a rectangle in which the length of each diagonal is 12, and one of the angles formed by the diagonal and a side measures 30°.

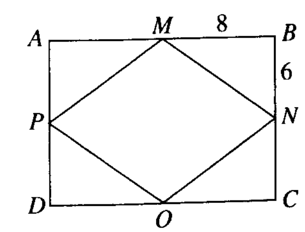
6. What is the area of the rectangle?
   - Ⓐ 18
   - Ⓑ 72
   - Ⓒ $18\sqrt{3}$
   - Ⓓ $36\sqrt{3}$
   - Ⓔ $36\sqrt{2}$

7. What is the perimeter of the rectangle?
   - Ⓐ 18
   - Ⓑ 24
   - Ⓒ $12 + 12\sqrt{3}$
   - Ⓓ $18 + 6\sqrt{3}$
   - Ⓔ $24\sqrt{2}$

8. How many sides does a polygon have if the measure of each interior angle is 8 times the measure of each exterior angle?
   - Ⓐ 8
   - Ⓑ 9
   - Ⓒ 10
   - Ⓓ 12
   - Ⓔ 18

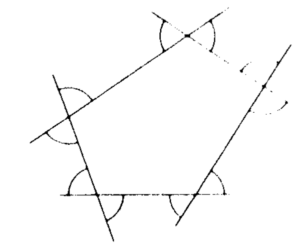
9. The length of a rectangle is 5 more than the side of a square, and the width of the rectangle is 5 less than the side of the square. If the area of the square is 45, what is the area of the rectangle?
   - Ⓐ 20
   - Ⓑ 25
   - Ⓒ 45
   - Ⓓ 50
   - Ⓔ 70

Questions 10–11 refer to the following figure, in which *M*, *N*, *O*, and *P* are the midpoints of the sides of rectangle *ABCD*.



10. What is the perimeter of quadrilateral *MNOP*?
   - Ⓐ 24
   - Ⓑ 32
   - Ⓒ 40
   - Ⓓ 48
   - Ⓔ 60

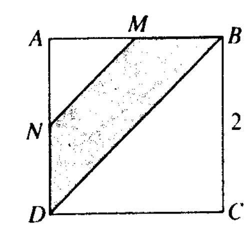
11. What is the area of quadrilateral *MNOP*?



12. In the figure above, what is the sum of the measures of all of the marked angles?
   - Ⓐ 360
   - Ⓑ 540
   - Ⓒ 720
   - Ⓓ 900
   - Ⓔ 1080

13. In quadrilateral *WXYZ*, the measure of angle *Z* is 10 more than twice the average of the measures of the other three angles. What is the measure of angle *Z*?
   - Ⓐ 100
   - Ⓑ 105
   - Ⓒ 120
   - Ⓓ 135
   - Ⓔ 150

Questions 14–15 refer to the following figure, in which *M* and *N* are the midpoints of two of the sides of square *ABCD*.



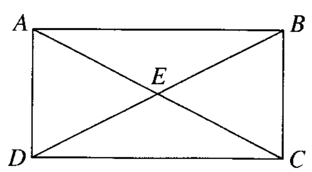
14. What is the perimeter of the shaded region?
   - Ⓐ 3
   - Ⓑ $2 + 3\sqrt{2}$
   - Ⓒ $3 + 2\sqrt{2}$
   - Ⓓ 5
   - Ⓔ 8

15. What is the area of the shaded region?

   Ⓐ 1.5
   Ⓑ 1.75
   Ⓒ 3
   Ⓓ $2\sqrt{2}$
   Ⓔ $3\sqrt{2}$

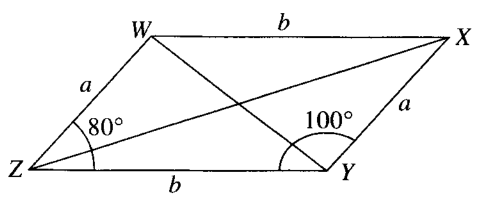
Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.



ABCD is a rectangle.

	Quantity A	Quantity B
16.	The area of $\triangle AED$	The area of $\triangle EDC$

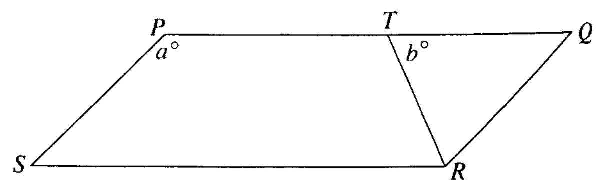


WXYZ is a parallelogram.

	Quantity A	Quantity B
17.	Diagonal *WY*	Diagonal *XZ*

	Quantity A	Quantity B
18.	The perimeter of a 30-60-90 right triangle whose shorter leg is $2x$	The perimeter of an octagon, each of whose sides is x

	Quantity A	Quantity B
19.	The perimeter of a rectangle whose area is 50	28



In parallelogram *PQRS*, *TR* bisects $\angle QRS$.

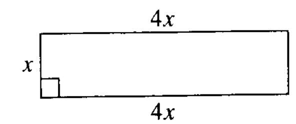
	Quantity A	Quantity B
20.	a	$2b$

ANSWER KEY

1. **60**	5. **C**	9. **A**	13. **150**	17. **B**
2. **B**	6. **D**	10. **C**	14. **B**	18. **A**
3. **D**	7. **C**	11. **96**	15. **A**	19. **A**
4. **C**	8. **E**	12. **C**	16. **C**	20. **C**

Answer Explanations

1. **60** Draw a diagram and label it.



Since the area is 144, then $144 = (4x)(x) = 4x^2 \Rightarrow x^2 = 36 \Rightarrow x = 6$.
So the width is 6, the length is 24, and the perimeter is 60.

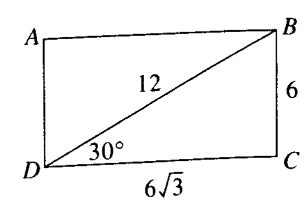
2. **(B)** The area of the square is $2^2 = 4$, and each triangle is one-fourth of the square. So the area of $\triangle DEC$ is 1.

3. **(D)** $\triangle DEC$ is a 45-45-90 right triangle whose hypotenuse, *DC*, is 2.
   Therefore, each of the legs is $\frac{2}{\sqrt{2}} = \sqrt{2}$. So the perimeter is $2 + 2\sqrt{2}$.

4. **(C)** The sum of the angles of a five-sided polygon is $(5 - 2) \times 180 = 3 \times 180 = 540$. Therefore, $540 = 2x + 3x + 3x + 5x + 5x = 18x$.
   So, $x = 540 \div 18 = 30$.
   The measure of the smallest angle is $2x = 2 \times 30 = 60$.

5. **(C)** The area of rectangle *ABCD* is $(x + 1)(x + 4) = x^2 + 5x + 4$.
   The area of rectangle *EFGH* is $(x + 2)(x + 3) = x^2 + 5x + 6$, which is exactly 2 more than the area of rectangle *ABCD*: $100 + 2 = 102$.

6. **(D)** Draw a picture and label it.



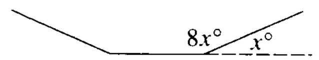
Since $\triangle BCD$ is a 30-60-90 right triangle, *BC* is 6 (half the hypotenuse) and *CD* is $6\sqrt{3}$.

So the area is $\ell w = 6(6\sqrt{3}) = 36\sqrt{3}$.

7. **(C)** The perimeter of the rectangle is $2(\ell + w) = 2(6 + 6\sqrt{3}) = 12 + 12\sqrt{3}$.

8. **(E)** The sum of the degree measures of an interior and exterior angle is 180, so $180 = 8x + x = 9x \Rightarrow x = 20$.

Since the sum of the measures of all the exterior angles is 360, there are $360 \div 20 = 18$ angles and 18 sides.



9. **(A)** Let x represent the side of the square. Then the dimensions of the rectangle are $(x + 5)$ and $(x - 5)$, and its area is $(x + 5)(x - 5) = x^2 - 25$. Since the area of the square is 45, $x^2 = 45 \Rightarrow x^2 - 25 = 45 - 25 = 20$.

10. **(C)** Each triangle surrounding quadrilateral *MNOP* is a 6-8-10 right triangle. So each side of *MNOP* is 10, and its perimeter is 40.

11. **96** The area of each of the triangles is $\frac{1}{2}(6)(8) = 24$, so together the four triangles have an area of 96. The area of the rectangle is $16 \times 12 = 192$. Therefore, the area of quadrilateral *MNOP* is $192 - 96 = 96$.
Note: Joining the midpoints of the four sides of *any* quadrilateral creates a parallelogram whose area is one-half the area of the original quadrilateral.

12. **(C)** Each of the 10 marked angles is an exterior angle of the pentagon. If we take one angle at each vertex, the sum of those five angles is 360; the sum of the other five is also 360: $360 + 360 = 720$.

13. **150** Let *W*, *X*, *Y*, and *Z* represent the measures of the four angles. Since $W + X + Y + Z = 360$, $W + X + Y = 360 - Z$. Also,

$$Z = 10 + 2\left(\frac{W + X + Y}{3}\right) = 10 + 2\left(\frac{360 - Z}{3}\right).$$

So $Z = 10 + \frac{2}{3}(360) - \frac{2}{3}Z = 10 + 240 - \frac{2}{3}Z \Rightarrow \frac{5}{3}Z = 250 \Rightarrow Z = 150$.

14. **(B)** Since *M* and *N* are midpoints of sides of length 2, *AM*, *MB*, *AN*, and *ND* are all 1. $MN = \sqrt{2}$, since it's the hypotenuse of an isosceles right triangle whose legs are 1; and $BD = 2\sqrt{2}$, since it's the hypotenuse of an isosceles right triangle whose legs are 2. So the perimeter of the shaded region is $1 + \sqrt{2} + 1 + 2\sqrt{2} = 2 + 3\sqrt{2}$.

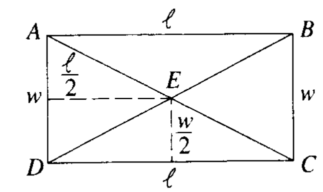
15. **(A)** The area of $\triangle ABD = \frac{1}{2}(2)(2) = 2$, and the area of $\triangle AMN$ is $\frac{1}{2}(1)(1) = 0.5$.

So the area of the shaded region is $2 - 0.5 = 1.5$.

16. **(C)** The area of $\triangle AED$ is $\frac{1}{2}w\left(\frac{\ell}{2}\right) = \frac{\ell w}{4}$.

The area of $\triangle EDC$ is $\frac{1}{2}\ell\left(\frac{w}{2}\right) = \frac{\ell w}{4}$.

Note: Each of the four small triangles has the same area.

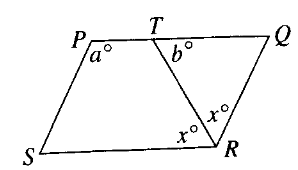


17. **(B)** By KEY FACT J5, since $\angle Z$ is acute and $\angle Y$ is obtuse, $(WY)^2 < a^2 + b^2$, whereas $(XZ)^2 > a^2 + b^2$.

18. **(A)** Since an octagon has eight sides, Quantity B is $8x$.
Quantity A: By KEY FACT J10, the hypotenuse of the triangle is $4x$, and the longer leg is $2x\sqrt{3}$. So the perimeter is $2x + 4x + 2x\sqrt{3}$. Since $\sqrt{3} > 1$, then $2x + 4x + 2x\sqrt{3} > 2x + 4x + 2x = 8x$.

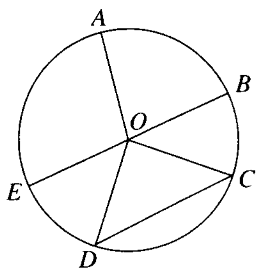
19. **(A)** The perimeter of a rectangle of area 50 can be as large as we like, but the least it can be is when the rectangle is a square. In that case, each side is $\sqrt{50}$, which is greater than 7, and so the perimeter is greater than 28.

20. **(C)** *TR* is a transversal cutting the parallel sides *PQ* and *RS*. So $b = x$ and $2b = 2x$. But since the opposite angles of a parallelogram are equal, $a = 2x$. So $a = 2b$.



11-L. CIRCLES

A ***circle*** consists of all the points that are the same distance from one fixed point called the ***center***. That distance is called the ***radius*** of the circle. The figure below is a circle of radius 1 unit whose center is at the point *O*. *A*, *B*, *C*, *D*, and *E*, which are each 1 unit from *O*, are all points on circle *O*. The word *radius* is also used to represent any of the line segments joining the center and a point on the circle. The plural of ***radius*** is ***radii***. In circle *O*, below, *OA*, *OB*, *OC*, *OD*, and *OE* are all radii. If a circle has radius *r*, each of the radii is *r* units long.

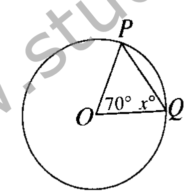


KEY FACT L1

Any triangle, such as $\triangle COD$ in the figure above, formed by connecting the endpoints of two radii, is isosceles.

EXAMPLE 1

If *P* and *Q* are points on circle *O*, what is the value of *x*?



SOLUTION.
Since $\triangle POQ$ is isosceles, angles *P* and *Q* have the same measure. Then, $70 + x + x = 180 \Rightarrow 2x = 110 \Rightarrow x = \mathbf{55}$.

A line segment, such as *CD* in circle *O* at the beginning of this section, both of whose endpoints are on a circle is called a ***chord***. A chord such as *BE*, which passes through the center of the circle, is called a ***diameter***. Since *BE* is the sum of two radii, *OB* and *OE*, it is twice as long as a radius.

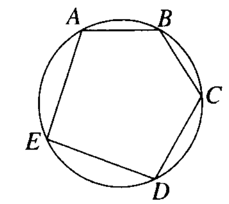
KEY FACT L2

If *d* is the diameter and *r* the radius of a circle, $d = 2r$.

KEY FACT L3

A diameter is the longest chord that can be drawn in a circle.

EXAMPLE 2



The radius of the circle is 0.1.

Quantity A	Quantity B
AB + *BC* + *CD* + *DE* + *EA*	1

SOLUTION.
Since the radius of the circle is 0.1, the diameter is 0.2. Therefore, the length of each of the five sides of pentagon *ABCDE* is less than 0.2, and the sum of their lengths is less than $5 \times 0.2 = 1$. The answer is **B**.

The total length around a circle, from *A* to *B* to *C* to *D* to *E* and back to *A*, is called the ***circumference*** of the circle. In every circle the ratio of the circumference to the diameter is exactly the same and is denoted by the symbol π (the Greek letter "pi").

KEY FACT L4

- $\pi = \dfrac{\text{circumference}}{\text{diameter}} = \dfrac{C}{d}$
- $C = \pi d$
- $C = 2\pi r$

KEY FACT L5

The value of π is approximately 3.14.

On GRE questions that involve circles, you are almost always expected to leave your answer in terms of π. So *don't* multiply by 3.14 until the final step, and then only if you have to. If you are ever stuck on a problem whose answers involve π, use your calculator to evaluate the answer or to test the answers. For example, assume that you think that an answer is about 50, and the answer choices are 4π, 6π, 12π, 16π, and 24π. Since π is slightly greater than 3, these choices are a little greater than 12, 18, 36, 48, and 72. The answer must be 16π. (To the nearest hundredth, 16π is actually 50.27, but approximating it by 48 was close enough.)

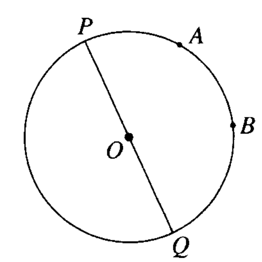
EXAMPLE 3

Quantity A	Quantity B
The circumference of a circle whose diameter is 12	The perimeter of a square whose side is 12

SOLUTION.
Quantity A: $C = \pi d = \pi(12)$. Quantity B: $P = 4s = 4(12)$.

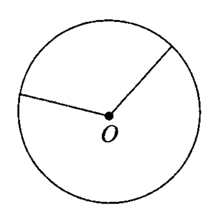
Since $4 > \pi$, Quantity **B** is greater. (Note: $12\pi = 12(3.14) = 37.68$, but *you should not have wasted any time calculating this.*)

An ***arc*** consists of two points on a circle and all the points between them. On the GRE, *arc AB* always refers to the smaller arc joining A and B.



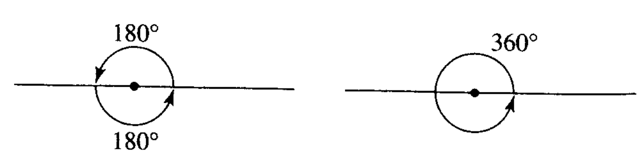
If we wanted to refer to the large arc going from A to B through P and Q, we would refer to it as arc APB or arc AQB. If two points, such as P and Q in circle O, are the endpoints of a diameter, they divide the circle into two arcs called ***semicircles***.

An angle whose vertex is at the center of a circle is called a ***central angle***.



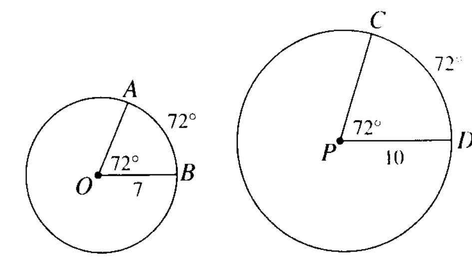
KEY FACT L6

The degree measure of a complete circle is 360°.



KEY FACT L7

The degree measure of an arc equals the degree measure of the central angle that intercepts it.



CAUTION

Degree measure is *not* a measure of length. In the circles above, arc *AB* and arc *CD* each measure 72°, even though arc *CD* is much longer.

How long *is* arc CD? Since the radius of Circle P is 10, its diameter is 20, and its circumference is 20π. Since there are 360° in a circle, arc CD is $\frac{72}{360}$, or $\frac{1}{5}$, of the circumference: $\frac{1}{5}(20\pi) = 4\pi$.

KEY FACT L8

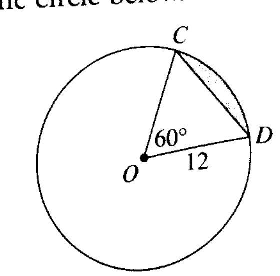
The formula for the area of a circle of radius r is $A = \pi r^2$.

The area of Circle P, below KEY FACT L7, is $\pi(10)^2 = 100\pi$ square units. The area of sector CPD is $\frac{1}{5}$ of the area of the circle: $\frac{1}{5}(100\pi) = 20\pi$.

KEY FACT L9

If an arc measures $x°$, the length of the arc is $\frac{x}{360}(2\pi r)$, and the area of the sector formed by the arc and 2 radii is $\frac{x}{360}(\pi r^2)$.

Examples 4 and 5 refer to the circle below.



EXAMPLE 4

What is the area of the shaded region?

Ⓐ $144\pi - 144\sqrt{3}$

Ⓑ $144\pi - 36\sqrt{3}$

Ⓒ $144 - 72\sqrt{3}$

Ⓓ $24\pi - 36\sqrt{3}$

Ⓔ $24\pi - 72\sqrt{3}$

SOLUTION.
The area of the shaded region is equal to the area of sector COD minus the area of $\triangle COD$. The area of the circle is $\pi(12)^2 = 144\pi$.

- Since $\frac{60}{360} = \frac{1}{6}$, the area of sector COD is $\frac{1}{6}(144\pi) = 24\pi$.
- Since $m\angle O = 60°$, $m\angle C + \angle D = 120°$ and since $\triangle COD$ is isosceles, $m\angle C = m\angle D$. So, they each measure 60°, and the triangle is equilateral.
- By KEY FACT J15, area of $\triangle COD = \frac{12^2\sqrt{3}}{4} = \frac{144\sqrt{3}}{4} = 36\sqrt{3}$. So the area of the shaded region is $\mathbf{24\pi - 36\sqrt{3}}$ **(D)**.

EXAMPLE 5

What is the perimeter of the shaded region?

Ⓐ $12 + 4\pi$

Ⓑ $12 + 12\pi$

Ⓒ $12 + 24\pi$

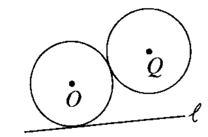
Ⓓ $12\sqrt{2} + 4\pi$

Ⓔ $12\sqrt{2} + 24\pi$

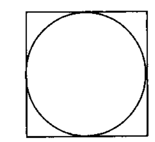
SOLUTION.
Since $\triangle COD$ is equilateral, $CD = 12$. Since the circumference of the circle is $2\pi(12) = 24\pi$, arc $CD = \frac{1}{6}(24\pi) = 4\pi$. So the perimeter is $\mathbf{12 + 4\pi}$ **(A)**.

Suppose that in Example 5 you see that $CD = 12$, but you don't remember how to find the length of arc CD. From the diagram, it is clear that it is slightly longer than CD, say 13. So you know that the perimeter is *about* 25. Now, mentally, using 3 for π, or with your calculator, using 3.14 for π, approximate the value of each of the choices and see which one is closest to 25. Only Choice A is even close.

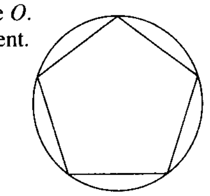
A line and a circle or two circles are ***tangent*** if they have only one point of intersection. A circle is ***inscribed*** in a triangle or square if it is tangent to each side. A polygon is ***inscribed*** in a circle if each vertex is on the circle.



Line ℓ is tangent to circle O.
Circles O and Q are tangent.



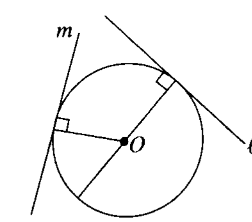
The circle is inscribed in the square.



The pentagon is inscribed in the circle.

KEY FACT L10

If a line is tangent to a circle, a radius (or diameter) drawn to the point where the tangent touches the circle is perpendicular to the tangent line.



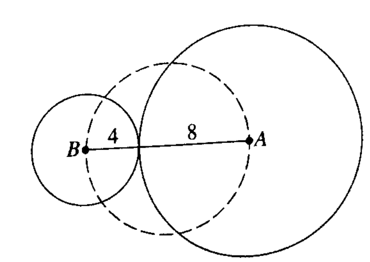
Lines ℓ and m are tangent to circle O.

EXAMPLE 6

A is the center of a circle whose radius is 8, and B is the center of a circle whose diameter is 8. If these two circles are tangent to one another, what is the area of the circle whose diameter is AB?

Ⓐ 12π Ⓑ 36π Ⓒ 64π Ⓓ 144π Ⓔ 256π

SOLUTION.
Draw a diagram.



Since the diameter, AB, of the dotted circle is 12, its radius is 6, and its area is $\pi(6)^2 = \mathbf{36\pi}$ **(B)**.

Practice Exercises — Circles

Discrete Quantitative Questions

1. What is the circumference of a circle whose area is 100π?

Ⓐ 10
Ⓑ 20
Ⓒ 10π
Ⓓ 20π
Ⓔ 25π

2. What is the area of a circle whose circumference is π?

Ⓐ $\frac{\pi}{4}$
Ⓑ $\frac{\pi}{2}$
Ⓒ π
Ⓓ 2π
Ⓔ 4π

3. What is the area of a circle that is inscribed in a square of area 2?

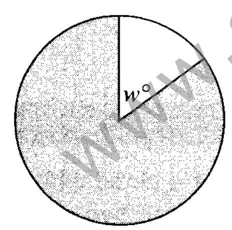
Ⓐ $\frac{\pi}{4}$
Ⓑ $\frac{\pi}{2}$
Ⓒ π
Ⓓ $\pi\sqrt{2}$
Ⓔ 2π

4. A square of area 2 is inscribed in a circle. What is the area of the circle?

Ⓐ $\frac{\pi}{4}$
Ⓑ $\frac{\pi}{2}$
Ⓒ π
Ⓓ $\pi\sqrt{2}$
Ⓔ 2π

5. A 5×12 rectangle is inscribed in a circle. What is the radius of the circle?

6. If, in the figure below, the area of the shaded sector is 85% of the area of the entire circle, what is the value of w?

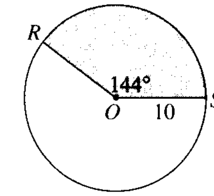


Ⓐ 15
Ⓑ 30
Ⓒ 45
Ⓓ 54
Ⓔ 60

7. The circumference of a circle is $a\pi$ units, and the area of the circle is $b\pi$ square units. If $a = b$, what is the radius of the circle?

Ⓐ 1
Ⓑ 2
Ⓒ 3
Ⓓ π
Ⓔ 2π

Questions 8–9 refer to the following figure.

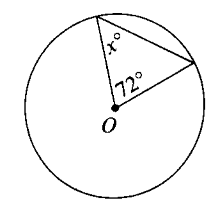


8. What is the length of arc RS?

Ⓐ 8
Ⓑ 20
Ⓒ 8π
Ⓓ 20π
Ⓔ 40π

9. What is the area of the shaded sector?

Ⓐ 8
Ⓑ 20
Ⓒ 8π
Ⓓ 20π
Ⓔ 40π



10. In the figure above, what is the value of x?

11. If A is the area and C the circumference of a circle, which of the following is an expression for A in terms of C?

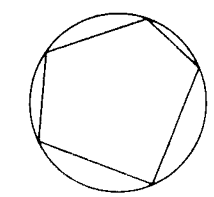
Ⓐ $\frac{C^2}{4\pi}$
Ⓑ $\frac{C^2}{4\pi^2}$
Ⓒ $2C$
Ⓓ $2C^2\sqrt{\pi}$
Ⓔ $\frac{C^2\sqrt{\pi}}{4}$

12. What is the area of a circle whose radius is the diagonal of a square whose area is 4?

Ⓐ 2π
Ⓑ $2\pi\sqrt{2}$
Ⓒ 4π
Ⓓ 8π
Ⓔ 16π

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.



	Quantity A	Quantity B
13.	The perimeter of the pentagon	The circumference of the circle

The circumference of a circle is C inches. The area of the same circle is A square inches.

	Quantity A	Quantity B
14.	$\frac{C}{A}$	$\frac{A}{C}$

	Quantity A	Quantity B
15.	The area of a circle of radius 2	The area of a semicircle of radius 3

C is the circumference of a circle of radius r

	Quantity A	Quantity B
16.	$\frac{C}{r}$	6